DIFFERENTIAL FORMS ON SYMPLECTIC QUOTIENTS

JORDAN ALAN WATTS

Abstract. We show that the de Rham complex defined by Sjamaar on a symplectic quotient is intrinsic. As a result, when obtaining a quotient via reduction in stages, the de Rham complex on the quotient is independent of the reduction.

Contents

1. Introduction

Let G be a compact Lie group acting on a connected symplectic manifold (M,ω) such that the action is Hamiltonian with proper momentum map Φ. In [29], Sjamaar introduces a notion of differential form on the symplectic reduced space $M/\!/_0\,G := \Phi^{-1}(0)/G$. In general the reduced space is not a manifold. In fact, it is a stratified space with symplectic strata whose ring of smooth functions carries a Poisson structure (see [14]).

In brief, $Z := \Phi^{-1}(0)$ is a G-invariant subset of M, and there exists a closed subgroup $K \leq G$ such that $Z_{(K)} := \{x \in Z \mid \text{there exists } g \in G \text{ satisfying } \mathrm{Stab}(x) = gKg^{-1}\}$ is an open dense subset of Z with its subspace topology. $Z_{(K)}$ is a G-manifold, and $Z_{(K)}/G$ is an open dense and connected subset of the symplectic quotient $M/\!/_0\,G = Z/G$. $Z_{(K)}/G$ is also a manifold with a (unique) smooth structure such that a function $f \in C^\infty(Z_{(K)}/G)$ if and only if $(\pi|_{Z_{(K)}})^*f$ locally extends to a smooth function on M. Let $i : Z \to M$ be the inclusion. A *differential k-form* μ on $M/\!/_0\,G$ is a differential k-form in $\Omega^k(Z_{(K)}/G)$ satisfying

$$(\pi|_{Z_{(K)}})^*\mu = (i|_{Z_{(K)}})^*\alpha$$

for some $\alpha \in \Omega^k(M)$. In particular, for $k = 0$, we recover the smooth structure $C^\infty(M/\!/_0\,G)$ introduced by Arms, Cushman and Gotay in [1]. These differential forms on $M/\!/_0\,G$ form a complex $(\Omega(M/\!/_0\,G), d)$ that satisfies a de Rham theorem, Poincaré Lemma, Stokes' theorem among other properties, as proven in [29].

Date: January 20, 2013.

The purpose of this paper is to prove that the complex $(\Omega(M/\!/_0 G), d)$ is intrinsic to $M/\!/_0 G$. That is, if (M', ω') was another connected symplectic manifold yielding a Hamiltonian G-action with symplectic quotient $M'/\!/_0 G$, then we have the following theorem.

Main Theorem. *If $F : M/\!/_0 G \to M'/\!/_0 G$ is a homeomorphism inducing an isomorphism $F^* : C^\infty(M'/\!/_0 G) \to C^\infty(M/\!/_0 G)$, then F restricts to a diffeomorphism from an open dense connected subset of $M/\!/_0 G$ to an open dense connected subset of $M'/\!/_0 G$ that induces an isomorphism of complexes $F^* : (\Omega(M'/\!/_0 G), d) \to (\Omega(M/\!/_0 G), d)$.*

As an application, we show that if the group G is a product $G = G_1 \times G_2$, then the resulting complex of differential forms does not depend on whether the reduction was done in stages or not, nor does it depend on the order of these stages. In [29], Sjamaar shows a special case of this.

The idea of the proof of the Main Theorem is as follows. The smooth structure on the reduced space makes it into a subcartesian space; that is, we can locally smoothly embed $M/\!/_0 G$ into Euclidean space. This structure induces Hamiltonian vector fields on $M/\!/_0 G$, which induce an orbit equal to the open dense connected subset of $M/\!/_0 G$ above. We then define a complex of differential forms as smooth fibrewise-linear functions on exterior products of the tangent vectors in the images of these vector fields. We then show that these are isomorphic to $(\Omega(M/\!/_0 G), d)$. Since these forms are intrinsic to $M/\!/_0 G$ equipped with $C^\infty(M/\!/_0 G)$, this will complete the proof.

Section 2 is a detailed set of preliminaries required to define $(\Omega(M/\!/_0 G), d)$ precisely, and to create the setting in which we will prove the Main Theorem. Section 3 reviews and develops the theory of vector fields on subcartesian spaces required to prove the Main Theorem. Section 4 defines differential forms as described above, and proves the Main Theorem. Section 5 is a review of reduction in stages, and the proof of the application mentioned above is found here.

Before continuing, it is worth mentioning that many different definitions of "smooth structure" appear in the literature. Stacey in his article [33] compares some of these. Stacey makes note that the smooth structures discussed are defined by maps into a set/topological space (*e.g.* diffeological spaces, introduced by Souriau; see [32] and [12]), maps out of a set/topological space (*e.g.* differential spaces, introduced by Sikorski; see [27] and [28]), or both (*e.g.* Frölicher spaces, introduced by Frölicher; see [10] and [22]). In this paper we will use the differential space approach: a differential space is a topological space X equipped with a sheaf of functions that satisfies certain axioms. However, we will still make use of maps into X inducing some form of structure. In fact, integral curves coming from local flows of vector fields play an essential role throughout. Structures such as stratifications and foliations translate into this language quite readily.

Finally, there has been much study of the quotient space $M/\!/_0 G$ in the case that it is not a manifold; this is generally referred to as *singular reduction*, and references include Arms, Cushman, and Gotay [1], Guillemin, Ginzburg, and Karshon [11], and Meinrenken and Sjamaar [20].

Acknowledgements: The author would like to thank Yael Karshon for her enthusiasm and many helpful conversations regarding this work, Reyer Sjamaar and Fatima Mahmood for their interest, and Jędrzej Śniatycki for some important discussions.

2. Preliminaries

The purpose of this section is to review concepts that will be used throughout the paper; in particular, so that the reader may understand precisely the main theorem and the setting in which we prove it.

The Setting. Let G be a compact Lie group acting smoothly on a connected smooth manifold M.

Definition 2.1. The *orbit space* M/G of this action is the set of equivalence classes given by the following relation on M: for $x, y \in M$, $x \sim y$ if x and y are in the same G-orbit; that is, if there exists $g \in G$ such that $g \cdot x = y$. We equip M/G with the quotient topology, which makes M/G into a Hausdorff and locally compact space.

Now assume M is a symplectic manifold with symplectic form ω.

Definition 2.2. A smooth G-action on (M, ω) is *Hamiltonian* if the action preserves ω and there exists a smooth map $\Phi : M \to \mathfrak{g}^*$ (where $\mathfrak{g}^*$ is the dual to the Lie algebra $\mathfrak{g}$ of G) satisfying:

(1) Φ is G-equivariant with respect to the coadjoint action of G on $\mathfrak{g}^*$,
(2) For any $\xi \in \mathfrak{g}$, let ξ_M be the vector field induced by ξ on M: for any $x \in M$,

$$\xi_M|_x := \frac{d}{dt}\Big|_{t=0} \exp(t\xi) \cdot x,$$

and let $\Phi^\xi(x) := \langle \Phi(x), \xi \rangle$, where $\langle , \rangle$ is the pairing between $\mathfrak{g}^*$ and $\mathfrak{g}$. Then

$$\xi_M \lrcorner\, \omega = -d\Phi^\xi.$$

We call Φ a *momentum map.*

Let $Z := \Phi^{-1}(0)$.

Theorem 2.3 (Marsden-Weinstein, Meyer)**.** *If 0 is a regular value of Φ, then $i : Z \hookrightarrow M$ is a smooth embedded G-invariant submanifold of M. If in addition G acts freely on Z, then the orbit space $M/\!/_0\, G := Z/G$ is a smooth manifold, $\pi_Z : Z \to M/\!/_0\, G$ is a principal G-bundle, and $M/\!/_0\, G$ admits a symplectic form ω_0 satisfying $\pi_Z^*\omega_0 = i^*\omega$.*

Proof. (See [16] and [21].) $\square$

Definition 2.4. $M/\!/_0\, G$ above is called the *symplectic reduced space* of the action, and ω_0 the *reduced symplectic form.*

If 0 is not a regular value of Φ, then Z may not be a smooth submanifold, and thus $M/\!/_0\, G$ need not be a smooth manifold. The latter case may also occur if G does not act freely on Z. In such cases, Z is still G-invariant, and the quotient space Z/G equipped with the quotient topology is still Hausdorff and locally compact. We thus obtain the following commutative diagramme of continuous maps, where $j : Z/G \hookrightarrow M/G$ is the inclusion, and i is smooth:

$$\begin{tikzcd} Z \arrow[r, "i"] \arrow[d, "\pi_Z"'] & M \arrow[d, "\pi"] \\ Z/G \arrow[r, "j"'] & M/G \end{tikzcd} \tag{1}$$

Stratified Spaces. We quickly review the definition of a decomposed space and a stratified space, and apply it to the situation above. (See [23] for details on decomposed and stratified spaces.)

Definition 2.5. A *decomposed space* X is a paracompact, second-countable, Hausdorff topological space, equipped with a locally finite partition $\mathcal{D}$ of X into locally closed subsets, called *pieces*, which with their subspace topologies admit smooth structures, and that satisfy the "frontier condition": for any $P_1, P_2 \in \mathcal{D}$, if $P_1 \cap \overline{P_2} \neq \emptyset$, then $P_1 \subset \overline{P_2}$. In this case P_1 is part of the boundary of P_2.

Remark 2.6. Throughout this paper, we will be concerned with subsets of $\mathbb{R}^n$, or topological spaces locally homeomorphic to subsets of $\mathbb{R}^n$, in which case "locally closed" and "locally compact" are equivalent.

Definition 2.7. A *stratified space* X is a paracompact, second-countable, Hausdorff topological space equipped with a map (called a *stratification*) sending $x \in X$ to the set germ S_x of a closed set $S \subseteq X$ at x. This map is required to satisfy: for any $x \in X$ there exists an open neighbourhood U of x and decomposition $\mathcal{D}_U$ of U such that for each $y \in U$, the set germ S_y at y is the set germ of the piece in $\mathcal{D}_U$ containing y.

Remark 2.8. Every decomposed space X has an induced stratification: for any $x \in X$, assign to x the set germ S_x of the piece containing x. The pieces of the decomposition in this case are called *strata*.

Example 2.9. Consider the square $[0,1] \times [0,1]$. The partition given by

$$\mathcal{D} = \{\{(0,0)\}, \{(0,1)\}, \{(1,0)\}, \{(1,1)\}, (0,1) \times \{0\},$$
$$(0,1) \times \{1\}, \{0\} \times (0,1), \{1\} \times (0,1), (0,1)^2\}$$

makes the square into a decomposed space, and with its induced stratification, a stratified space.

Definition 2.10. Let $(X, \mathcal{S}_X)$ and $(Y, \mathcal{S}_Y)$ be stratified spaces. A continuous map $f : X \to Y$ is a *stratified map* if for any $x \in X$ there exist an open neighbourhood $V \subseteq Y$ of $f(x)$ and an open neighbourhood $U \subseteq f^{-1}(V)$ of x, decompositions $\mathcal{D}_U$ of U and $\mathcal{D}_V$ of V inducing the restrictions of $\mathcal{S}_X$ to U and $\mathcal{S}_Y$ to V, respectively, so that for any $y \in U$, there is an open neighbourhood $W \subseteq U$ of y satisfying $f|_{(S_X)_y \cap W}$ is a smooth map into $(S_Y)_{f(y)}$. $(S_X)_y$ and $(S_y)_{f(y)}$ here are the pieces in $\mathcal{D}_U$ and $\mathcal{D}_V$ containing y and $f(y)$, respectively. An easier formulation of this is to say that f maps the set germ $(\mathcal{S}_X)_x$ onto a subgerm of $(\mathcal{S}_Y)_{f(x)}$.

Remark 2.11. In the case of decomposed spaces $(X, \mathcal{D}_X)$ and $(Y, \mathcal{D}_Y)$ with their induced stratifications, the above stratified map reduces to mapping each stratum of $\mathcal{D}_X$ into a stratum of $\mathcal{D}_Y$ smoothly.

Definition 2.12. Fix a topological space X with decompositions $\mathcal{D}_1$ and $\mathcal{D}_2$. $\mathcal{D}_1$ is a *refinement* of $\mathcal{D}_2$, denoted $\mathcal{D}_1 \geq \mathcal{D}_2$ if for every piece $P_1 \in \mathcal{D}_1$, there exists $P_2 \in \mathcal{D}_2$ such that $P_1 \subseteq P_2$. This induces a partial ordering on the set of decompositions on X. We say that $\mathcal{D}$ is *minimal* if for any $\mathcal{D}'$ such that $\mathcal{D} \geq \mathcal{D}'$, we have $\mathcal{D} = \mathcal{D}'$.

Example 2.13. The square $[0,1]^2$ with the decomposition given in Example 2.9 is minimal.

We now return to the setting above in Diagramme 1: G acts on (M, ω) in a Hamiltonian fashion with momentum map Φ, Z is the G-invariant level set $\Phi^{-1}(0)$, and Z/G is the quotient equipped with the quotient topology. Let H be a closed subgroup of G, and let $M_{(H)}$ be the set of all points in M whose stabiliser is a conjugate of H. Then, M is the disjoint union of the sets $M_{(H)}$ as H runs over closed subgroups of G.

The quotient map $\pi : M \to M/G$ partitions M/G into sets $\pi(M_{(H)})$ as H runs over closed subgroups of G. The sets $Z_{(H)} := M_{(H)} \cap Z$ form a partition of Z, and this in turn induces a partition on Z/G via π_Z similar to that on M/G.

Theorem 2.14 (Principal Orbit Theorem)**.**

(1) *The partitions on M, Z and the corresponding orbit spaces M/G and Z/G defined above induce stratifications, where the strata are given by connected components of the sets in the partitions.*
(2) *Each stratum in the stratification on M and Z is G-invariant.*
(3) *There exists a closed subgroup K_1 of G such that $M_{(K_1)}$ is an open dense stratum, and hence $\pi(M_{(K_1)})$ is an open dense stratum of M/G. Also, if Φ is a proper map, there exists a closed subgroup K_2 of G such that $Z_{(K_2)}$ is an open dense stratum, and $\pi_Z(Z_{(K_2)})$ is an open dense connected stratum.*
(4) *The maps i, π, π_Z and j are stratified with respect to the stratifications described above.*

Proof. The last statement above is clear by definition of the decompositions. See [9] and [14] for the first three statements. □

Definition 2.15. We call the above stratifications *orbit-type stratifications* of each respective space. We will denote $i|_{Z_{(H)}}$ by $i_{(H)}$ and $\pi_Z|_{Z_{(H)}}$ by $\pi_{(H)}$ for each relevant H.

Theorem 2.16 (Bierstone)**.** *The orbit-type stratification of M/G is minimal.*

Proof. See [5] and [6]. □

To summarise, Diagramme 1 sits in the category of stratified spaces.

Differential Structures. We review the notions of differential and subcartesian spaces, and then apply it to our setting above.

Definition 2.17 (Sikorski: [27], [28])**.** Let X be a topological space. A *differential structure*, sometimes called a *Sikorski structure*, on X is a family $\mathcal{F}$ of continuous functions into $\mathbb{R}$ satisfying:

(1) $\{f^{-1}((a, b)) \mid f \in \mathcal{F} \text{ and } (a, b) \subseteq \mathbb{R}\}$ is a subbasis for the topology on X.
(2) For any open set $U \subseteq X$, positive integer k, functions $f_1, ..., f_k \in \mathcal{F}$ and $F \in C^\infty(\mathbb{R}^k)$, the composition $F(f_1, ..., f_k)$ is contained in $\mathcal{F}$.
(3) Let $f : X \to \mathbb{R}$ be a function such that for any $x \in X$ there exists an open neighbourhood $U \subseteq X$ of x and a function $g \in \mathcal{F}$ satisfying $f|_U = g|_U$. Then $f \in \mathcal{F}$.

A set X equipped with a differential structure $\mathcal{F}$ is called a *differential space* or a *Sikorski space*, and denoted $(X, \mathcal{F})$.

Definition 2.18. Let $(X, \mathcal{F}_X)$ and $(Y, \mathcal{F}_Y)$ be two differential spaces. A continuous map $F : X \to Y$ is *(Sikorski) smooth* if $F^*\mathcal{F}_Y \subseteq \mathcal{F}_X$. F is called a *diffeomorphism* if it is smooth and has a smooth inverse.

Differential spaces along with smooth maps form a category. One of the most important properties of this category is given in the definition below.

Definition 2.19. Let $(X, \mathcal{F})$ be a differential space, and let $Y \subseteq X$ be any subset. Then Y, with the subspace topology induced by X, acquires a differential structure $\mathcal{F}_Y$ as follows: $f \in \mathcal{F}_Y$ if and only if for every $x \in Y$ there is an open neighbourhood $U \subseteq X$ of x and a function $\tilde{f} \in \mathcal{F}$ such that

$$f|_{U\cap Y} = \tilde{f}|_{U\cap Y}.$$

We call $(Y, \mathcal{F}_Y)$ a *differential subspace* of X.

Definition 2.20. Fix a differential space $(X, \mathcal{F})$ We say that $\mathcal{F}$ is *generated* by a family $\mathcal{Q} \subseteq \mathcal{F}$ if for any $f \in \mathcal{F}$ and $x \in X$ there exists an open neighbourhood $U \subseteq X$ of x, functions $q_1, ..., q_k \in \mathcal{Q}$ and a function $F : \mathbb{R}^k \to \mathbb{R}$ satisfying $f|_U = F(q_1, ..., q_k)|_U$.

Example 2.21. Let $(X, \mathcal{F})$ and $(Y, \mathcal{G})$ be two differential spaces. Then form the *product differential space* $(X \times Y, \mathcal{F} \times \mathcal{G})$ where $\mathcal{F} \times \mathcal{G}$ is generated by functions of the form $f \circ \mathrm{pr}_X$ for $f \in \mathcal{F}$ or $g \circ \mathrm{pr}_Y$ for $g \in \mathcal{G}$. Here, pr_X and pr_Y are the projections.

Definition 2.22. Let $(X, \mathcal{F})$ be a differential space, and let $\sim$ be an equivalence relation on X. Then $X/\sim$ obtains a differential structure $\mathcal{G} = \{f : X/\sim \to \mathbb{R} \mid \pi^* f \in \mathcal{F}\}$ where $\pi : X \to X/\sim$ is the quotient map.

Remark 2.23. Note that we do not endow the set $X/\sim$ above with the quotient topology. In general, the topology induced by $\mathcal{G}$ and the quotient topology do not match. The quotient map, however, is continuous (and hence smooth by definition of the differential structure on the quotient).

Definition 2.24. A *subcartesian space* is a paracompact, second-countable, Hausdorff differential space $(S, C^\infty(S))$ where for each $x \in S$ there is an open neighbourhood $U \subseteq S$ of x and a diffeomorphism $\varphi : U \to \tilde{U} \subseteq \mathbb{R}^n$, called a *chart*, onto $\tilde{U}$ as a differential subspace of $\mathbb{R}^n$.

Remark 2.25. The original definition of a subcartesian space was given by Aronszajn and Szeptycki (see [2], [3] and [4]). While the definition is different, it is in fact equivalent to the one above.

Remark 2.26.

(1) If R is a closed differential subspace of a subcartesian space S, then $C^\infty(R) = C^\infty(S)|_R$, the restrictions of functions in $C^\infty(S)$ to R.
(2) Note that $C^\infty(S)$ induces a sheaf on S: for any open subset $U \subseteq S$, $C^\infty(U)$ is the differential structure on U as a differential subspace of S.
(3) Subcartesian spaces, along with smooth maps between them, form a full subcategory of the category of differential spaces.
(4) A subcartesian space admits smooth partitions of unity (see [17]).

(5) For any subset $A \subseteq \mathbb{R}^n$, define $\mathfrak{n}(A)$ to be the ideal of all smooth functions on $\mathbb{R}^n$ whose restrictions to A vanish. Let S be a subcartesian space. Then, for each chart $\varphi : U \to \tilde{U} \subseteq \mathbb{R}^n$, the set of restrictions of functions in $C^\infty(\mathbb{R}^n)$ to $\tilde{U}$ is isomorphic to $C^\infty(\mathbb{R}^n)/\mathfrak{n}(\tilde{U})$. We thus have $\varphi^* C^\infty(\mathbb{R}^n) \cong C^\infty(\mathbb{R}^n)/\mathfrak{n}(\tilde{U})$.

Definition 2.27. Let $(X, \mathcal{F})$ be a differential space. $\mathcal{F}$ is *locally finitely generated* if for any $x \in X$, there is an open neighbourhood U of x and functions $q_1, ..., q_k \in \mathcal{F}$ such that for any $f \in \mathcal{F}$, there exists $F \in C^\infty(\mathbb{R}^k)$ satisfying $f|_U = F(q_1, ..., q_k)|_U$. $(X, \mathcal{F})$ is *locally finitely generated and separated* if U and $q_1, ..., q_k$ above can be chosen such that $q_1, ..., q_k$ separate all points in U.

Proposition 2.28. *Let $(X, \mathcal{F})$ be a paracompact, second-countable differential space where $\mathcal{F}$ is locally finitely generated and separated. Then $(X, \mathcal{F})$ is a subcartesian space. Conversely, any subcartesian space is locally finitely generated and separated.*

Proof. Fix $x \in X$. By the hypothesis, there exist an open neighbourhood U of x and $q_1, ..., q_n \in \mathcal{F}$ separating all points in U. Let $\varphi : U \to \mathbb{R}^n$ be defined by $\varphi = (q_1|_U, ..., q_n|_U)$. Then by definition of a differential structure, φ is a smooth map between U and $\mathbb{R}^n$. Since $f_1|_U, ..., f_n|_U$ separate points of U, φ is bijective.

Next, for any $f \in \mathcal{F}$, there exists $F \in C^\infty(\mathbb{R}^n)$ such that

$$f|_U = F(q_1, ..., q_n)|_U = F \circ \varphi.$$

Hence, $(\varphi^{-1})^* f = F|_U$, and so φ^{-1} is smooth.

Finally, since X is paracompact, there exists a locally finite covering of X with open sets $\{U_\alpha\}_{\alpha \in A}$ such that for each $\alpha \in A$, there exist $q_1^\alpha, ..., q_{n_\alpha}^\alpha \in \mathcal{F}$ that separate points in U_α. Let $\{\zeta_\alpha\}_{\alpha \in A}$ be a partition of unity subordinate to $\{U_\alpha\}_{\alpha \in A}$. Then, the set of functions $\bigcup_{\alpha \in A} \{\zeta_\alpha q_1^\alpha, ..., \zeta_\alpha q_{n_\alpha}^\alpha\}$ separates all points of X, and so X is Hausdorff.

For the converse, let S be a subcartesian space. For any $x \in S$, let $\varphi : U \to \tilde{U} \subseteq \mathbb{R}^n$ be a chart about x, and let $q^1, ..., q^n$ be the coordinate functions on $\mathbb{R}^n$. Then $\varphi^* q^1, ..., \varphi^* q^n$ generate the restriction of any function $f \in C^\infty(S)$ to U, and they separate all points in U. $\square$

This proposition was inspired by a similar one in [22].

Lemma 2.29. *Let R and S be subcartesian spaces, and let $F : R \to S$ be any map between them. Then F is smooth if and only if F is continuous and for every $x \in R$, there is a chart $\varphi : U \to \tilde{U} \subseteq \mathbb{R}^m$ about x and a chart $\psi : V \to \tilde{V} \subseteq \mathbb{R}^n$ about $F(x)$ such that $\psi \circ F \circ \varphi^{-1} : \tilde{U} \to \tilde{V}$ extends to a smooth map $\tilde{F} : \mathbb{R}^m \to \mathbb{R}^n$.*

Proof. Fix $x \in R$, and let $\psi : V \to \tilde{V} \subseteq \mathbb{R}^n$ be a chart about $F(x)$. Let $q^1, ..., q^n$ be coordinate functions on V; that is, $q^i := \psi^* x^i$ for $i = 1, ..., n$. Then, $q^i \in C^\infty(V)$, and so $(F|_{F^{-1}(V)})^* q^i \in C^\infty(F^{-1}(V))$. Let $\varphi : U \to \tilde{U} \subseteq \mathbb{R}^m$ be a chart about x such that $U \subseteq F^{-1}(V)$, and let $p^1, ..., p^n \in C^\infty(\mathbb{R}^m)$ such that $(F|_U)^* q^i = \varphi^* p^i$ (shrinking U if necessary). Define $\tilde{F} := (p^1, ..., p^n) \in C^\infty(\mathbb{R}^m, \mathbb{R}^n)$. Then,

$$\psi^{-1} \circ \tilde{F} \circ \varphi = \psi^{-1} \circ ((F|_U)^* q^1, ..., (F|_U)^* q^n) = F|_U.$$

Thus $\tilde{F}|_{\tilde{U}} = \psi \circ F \circ \varphi^{-1}$.

Conversely, let $f \in C^\infty(S)$. Then, for every $x \in R$, there exist a chart $\psi : V \to \tilde{V} \subseteq \mathbb{R}^n$ about $F(x)$, a chart $\varphi : U \to \tilde{U} \subseteq \mathbb{R}^m$ about x with $U \subseteq F^{-1}(V)$ and a smooth map $\tilde{F} : \mathbb{R}^m \to \mathbb{R}^n$ and a smooth function $\tilde{f} \in C^\infty(\mathbb{R}^n)$ such that $f|_V = \psi^* \tilde{f}$ and

$$\begin{aligned}(F^* f)|_U =& f \circ \psi^{-1} \circ \tilde{F} \circ \varphi \\ =& \tilde{f} \circ \tilde{F} \circ \varphi.\end{aligned}$$

But, $\tilde{f} \circ \tilde{F} \in C^\infty(\mathbb{R}^m)$ and so $\tilde{f} \circ \tilde{F} \circ \varphi = (F^* f)|_U \in \varphi^* C^\infty(\mathbb{R}^m)$. Hence, $F^* f \in C^\infty(R)$. Thus, F is also smooth. □

Equip M/G with the quotient differential structure, which we denote by $C^\infty(M/G)$. Note that $\pi^* : C^\infty(M/G) \to C^\infty(M)^G$ is an isomorphism of $\mathbb{R}$-algebras, where $C^\infty(M)^G$ is the algebra of G-invariant smooth functions.

Theorem 2.30. *If G is a compact Lie group acting on a connected smooth manifold M, then M/G is a subcartesian space and $\pi^* : C^\infty(M/G) \to C^\infty(M)^G$ is an isomorphism of $\mathbb{R}$-algebras.*

Proof. Cushman and Śniatycki have shown in [8] that the topology induced by $C^\infty(M/G)$ is equal to the quotient topology on M/G.

Recall the slice theorem of Koszul [13]: fix $x \in M$. Then if H is the stabiliser of x and V is the slice (or isotropy) representation at x, then there exist a G-invariant open neighbourhood U of x, a G-invariant neighbourhood W of $[e, 0] \in G \times_H V$ and a G-equivariant diffeomorphism $F : U \to W$ sending x to $[e, 0]$.

Next, shrink U so that it intersects only a finite number of orbit-type strata (which is possible since the orbit-type stratification is locally finite). A *proper embedding* $\varphi : X \to Y$ between differential spaces $(X, \mathcal{F})$ and $(Y, \mathcal{G})$ is a smooth, proper and injective map such that $\varphi^* : \mathcal{G} \to \mathcal{F}$ is surjective. Then by the Mostow-Palais theorem (see [7]), there exist $N \geq 0$ and a linear G-action on $\mathbb{R}^N$ and a G-equivariant proper embedding $\varphi : U \hookrightarrow \mathbb{R}^N$. φ descends to a proper injection $\tilde{\varphi} : U/G \hookrightarrow \mathbb{R}^N/G$.

Hilbert showed that $\mathbb{R}[x^1, ..., x^N]^G$ is a finitely generated algebra (see [19]). Let $\{p_1, ..., p_k\}$ be a set of generators of $\mathbb{R}[x^1, ..., x^N]^G$, and define $p = (p_1, ..., p_k) : \mathbb{R}^N \to \mathbb{R}^k$. Schwarz showed in [24] that p descends to a proper injective map $\tilde{p} : \mathbb{R}^N/G \hookrightarrow \mathbb{R}^k$. Thus, equipping $\mathbb{R}^N/G$ with the differential structure $C^\infty(\mathbb{R}^N/G) := \tilde{p}^* C^\infty(\mathbb{R}^k)$, $\tilde{p}$ becomes a proper embedding, $\mathbb{R}^N/G$ is diffeomorphic to a differential subspace of $\mathbb{R}^k$, and hence is subcartesian. By Schwarz [24], $C^\infty(\mathbb{R}^N)^G = p^* C^\infty(\mathbb{R}^k)$, and so if $\pi' : \mathbb{R}^N \to \mathbb{R}^N/G$ is the quotient map, then $\pi'^* : C^\infty(\mathbb{R}^N/G) \to C^\infty(\mathbb{R}^N)^G$ is a surjection. Since π' is surjective, then π^* is injective, and we get $\pi'^* : C^\infty(\mathbb{R}^N/G) \to C^\infty(\mathbb{R}^N)^G$ is an isomorphism of $\mathbb{R}$-algebras.

$$\begin{tikzcd}
M & U \arrow[l, hook'] \arrow[r, hook, "\varphi"] \arrow[d, "\pi|_U"'] & \mathbb{R}^N \arrow[r, "p"] \arrow[d, "\pi'"'] & \mathbb{R}^k \\
M/G & U/G \arrow[l, hook'] \arrow[r, hook, "\tilde{\varphi}"'] & \mathbb{R}^N/G \arrow[ur, "\tilde{p}"'] &
\arrow[from=1-1, to=2-1, "\pi"']
\end{tikzcd} \tag{2}$$

Since φ is a proper embedding, $\varphi^* : C^\infty(\mathbb{R}^N) \to C^\infty(U)$ is a surjection. Let $f \in C^\infty(U)^G$. Then there exists $g \in C^\infty(\mathbb{R}^N)^G$ such that $f = \varphi^* g$. $g|_{\varphi(U)}$ is G-invariant, and averaging over G, we may assume g is G-invariant everywhere on $\mathbb{R}^N$. Thus, φ^* restricts to a surjection $C^\infty(\mathbb{R}^N)^G \to C^\infty(U)^G$. Since $\pi|_U$ is a surjection from U to U/G, $(\pi|_U)^*$ is an injection. Fixing $f \in C^\infty(U)^G$, there is a function $g \in C^\infty(\mathbb{R}^N)^G$ such that $f = \varphi^* g$. But since $\pi'^* : C^\infty(\mathbb{R}^N)^G \to C^\infty(\mathbb{R}^N/G)$ is an isomorphism, there is some $g' \in C^\infty(\mathbb{R}^N/G)$ such that $\varphi^*(\pi^* g') = (\pi|_U)^*(\tilde{\varphi}^* g') = f$. Hence, $(\pi|_U)^*$ is an isomorphism.

For any $x \in M/G$, there exist a neighbourhood W of x and a proper embedding $\psi : W \hookrightarrow \mathbb{R}^k$. Let $q^1, ..., q^k$ be coordinate functions on $\mathbb{R}^k$. Let $f^i := (\pi|_{\pi^{-1}(W)})^* \psi^* q^i \in C^\infty(\pi^{-1}(W))$ for $i = 1, ..., k$, which are G-invariant on $\pi^{-1}(W)$. Extend these to smooth functions on M, and average them over G to obtain G-invariant functions $f^1, ..., f^k \in C^\infty(M)^G$ such that for each $i = 1, ..., k$, $f^i = \pi^* g^i$ where $g^i \in C^\infty(M/G)$. Note that $\{g^1, ..., g^k\}$ separates points in W and since $g^i|_W = q^i \circ \psi$, they generate restrictions of functions in $C^\infty(M/G)$ to W. Thus, $C^\infty(M/G)$ is locally finitely generated and separated, and since M/G is paracompact and second-countable, by Proposition 2.28, M/G is subcartesian. □

Let $C^\infty(Z/G)$ be the differential structure on Z/G induced by j.

Theorem 2.31. *Z/G is a subcartesian space, and $\pi_Z^* : C^\infty(Z/G) \to C^\infty(Z)^G$ is an isomorphism of $\mathbb{R}$-algebras.*

Proof. Note that Z/G is a closed subset of M/G (and hence is subcartesian), and so $C^\infty(Z/G) = C^\infty(M/G)|_{Z/G}$. We now show that π_Z is smooth. Let $f \in C^\infty(Z/G)$. Then there exists $g \in C^\infty(M/G)$ such that $f = j^* g$. There exists $\tilde{g} = \pi^* g \in C^\infty(M)^G$. Let $\tilde{f} = i^* \tilde{g} \in C^\infty(Z)^G$. Then, $\tilde{f} = \pi_Z^* f$.

Next, since π_Z is surjective, π_Z^* is injective. To show that π_Z^* is surjective, fix $\tilde{f} \in C^\infty(Z)^G$. Since Z is closed, there exists $\tilde{g} \in C^\infty(M)$ such that $\tilde{f} = i^* \tilde{g}$. Averaging over G, we may assume that $\tilde{g}$ is G-invariant. Thus, there exists $g \in C^\infty(M/G)$ such that $\pi^* g = \tilde{g}$. Thus, $f = j^* g \in C^\infty(Z/G)$, and $\pi_Z^* f = \tilde{f}$. We get that π_Z^* is an isomorphism of $\mathbb{R}$-algebras. □

Remark 2.32. The smooth structure $C^\infty(Z/G)$ is isomorphic to a smooth structure on Z/G introduced by Arms, Cushman and Gotay in [1]. The isomorphism in the theorem above is in fact its definition.

Basic Forms. We continue with the setting given by Diagramme 1; in particular, we have a compact connected Lie group G acting in a Hamiltonian fashion on a connected symplectic manifold (M, ω) with proper momentum map $\Phi : M \to \mathfrak{g}^*$. Let $i : Z := \Phi^{-1}(0) \hookrightarrow M$ be the (G-equivariant) inclusion. All four spaces in the diagramme come equipped with their orbit-type stratifications and smooth structures, and the maps of the diagramme are stratified and smooth.

Definition 2.33. A differential form $\mu \in \Omega^k(M)$ is *basic* if it is

(1) *G-invariant*: for any $g \in G$, $g^*\mu = \mu$, and
(2) *horizontal*: for any $\xi \in \mathfrak{g}$,
$$\xi_M \lrcorner \mu = 0.$$

Denote the set of all basic k-forms on M by $\Omega^k_{basic}(M)$.

Theorem 2.34 (Koszul)**.** *$(\Omega^*_{basic}(M), d)$ forms a subcomplex of $(\Omega^*(M), d)$, and the corresponding cohomology $H^*_{basic}(M)$ is isomorphic to the singular cohomology $H^*(M/G)$ with real coefficients.*

Proof. See [13]. □

Remark 2.35. If M/G is a smooth manifold, then for each $\mu \in \Omega^k_{basic}(M)$, there exists a unique $\eta \in \Omega^k(M/G)$ such that $\pi^*\eta = \mu$. Thus, in light of the above theorem and the de Rham theorem, $\pi^* : (\Omega^*(M/G), d) \to (\Omega^*_{basic}(M), d)$ is an isomorphism of complexes.

Left-Invariant and Hamiltonian Vector Fields.

Definition 2.36. A vector field X on M is *left-invariant* if for any $g \in G$, $g_*X = X$. Denote the set of left-invariant vector fields by $\mathrm{Vect}(M)^G$. A vector field X on M is *Hamiltonian* if there exists a function $f \in C^\infty(M)$ satisfying $X \lrcorner \omega = -df$. In this case, it is customary to denote X as X_f. This induces an $\mathbb{R}$-linear map $C^\infty(M) \to \mathrm{Vect}(M)$. Denote the image $\mathrm{Ham}(M)$, and the left-invariant Hamiltonian vector fields by $\mathrm{Ham}(M)^G$.

Remark 2.37. $\mathrm{Vect}(M)^G$, $\mathrm{Ham}(M)$ and $\mathrm{Ham}(M)^G$ are all Lie subalgebras of $\mathrm{Vect}(M)$ under the commutator bracket. Note also that for any $X \in \mathrm{Vect}(M)^G$, the local flow $(t, x) \mapsto \exp(tX)(x)$ of X is G-equivariant: for any $g \in G$,
$$g \cdot \exp(tX)(x) = \exp(tX)(g \cdot x).$$

Example 2.38. Let $\xi \in \mathfrak{g}$. Then the induced vector field ξ_M is Hamiltonian.

Proposition 2.39.

(1) *If X is a left-invariant vector field, then for any $H \leq G$ such that $M_{(H)}$ is nonempty, X is tangent to $M_{(H)}$, and so restricts to a vector field on $M_{(H)}$.*
(2) *If X is a left-invariant Hamiltonian vector field, then X is tangent to the G-manifold $Z_{(H)}$, and so $i^*_{(H)}X$ is well-defined.*

Proof.

(1) Let ψ_t be the flow of X. Then ψ_t is G-equivariant: if $x \in M$ and $g \in G$, then $\psi_t(g \cdot x) = g \cdot \psi_t(x)$ for all t in the flow domain. If $g \in \mathrm{Stab}(x) =: H$, then $\psi_t(x) = \psi_t(g \cdot x) = g \cdot \psi_t(x)$, and so ψ_t preserves stabilisers along their trajectories. Thus, these trajectories remain in $M_{(H)}$, and so X is tangent to $M_{(H)}$.
(2) Using the first statement of the proposition, it is enough to show that $X\Phi^\xi = 0$ for any $\xi \in \mathfrak{g}$. Let $f \in C^\infty(M)$ such that $X = X_f$. By averaging over G, we can choose f such that it is G-invariant. Thus,
$$\begin{aligned} X\Phi^\xi =& d\Phi^\xi(X) \\ =& \omega(X, \xi_M) \\ =& -df(\xi_M) = 0. \end{aligned}$$
This completes the proof.

□

Lemma 2.40. *For any $H \leq G$ such that $Z_{(H)}$ is nonempty, there is a Lie algebra homomorphism* $\mathrm{Ham}(M)^G \to \mathrm{Ham}((Z/G)_{(H)})$ *sending X_f to X_h where $h = j^*((\pi^*)^{-1} f)|_{(Z/G)_{(H)}}$, where $(\pi^*)^{-1} : C^\infty(M)^G \to C^\infty(M/G)$.*

Proof. Fix $X_f \in \mathrm{Ham}(M)^G$ where $f \in C^\infty(M)^G$ and $H \leq G$ such that $Z_{(H)}$ is nonempty. Then there exists $g \in C^\infty(M/G)$ such that $\pi^* g = f$. By Proposition 2.39, X_f restricts to a G-invariant vector field $(X_f)|_{Z_{(H)}}$ on $Z_{(H)}$. This descends via $\pi_{(H)}$ to a vector field Y on $(Z/G)_{(H)}$. We claim that Y is the Hamiltonian vector field of the smooth function $j^*g|_{(Z/G)_{(H)}}$. Indeed,

$$\begin{aligned}
\pi^*_{(H)}(Y \lrcorner\, \omega_{(H)}) =& X_f|_{Z_{(H)}} \lrcorner\, i^*_{(H)}\omega \\
=& i^*_{(H)}(X_f \lrcorner\, \omega) \\
=& i^*_{(H)}(-df) \\
=& (\pi \circ i_{(H)})^*(-dg) \\
=& (j \circ \pi_{(H)})^*(-dg) \\
=& \pi^*_{(H)}(-dj^*g|_{(Z/G)_{(H)}}).
\end{aligned}$$

By Theorem 2.34 and Remark 2.35, $\pi^*_{(H)}$ is an isomorphism onto its image, and so we conclude that $Y \lrcorner\, \omega_{(H)} = -dj^*g$. Letting $h = j^*g|_{(Z/G)_{(H)}}$, we thus have $Y = X_h$. □

For any $H \leq G$ such that $Z_{(H)}$ is nonempty, let $X_f, X_g \in \mathrm{Ham}(M)^G$. There exist $Y_1, Y_2 \in \mathrm{Ham}((Z/G)_{(H)})^G$ satisfying Lemma 2.40. We have the following second lemma.

Lemma 2.41. $i^*_{(H)}(\omega(X_f, X_g)) = \pi^*_{(H)}(\omega_{(H)}(Y_1, Y_2))$.

Proof. By Lemma 2.40, since $X_f \in \mathrm{Ham}^G(M)$, there exists $Y_1 \in \mathrm{Ham}((Z/G)_{(H)})$ which is Hamiltonian with respect to a function $h_1 \in C^\infty((Z/G)_{(H)})$ satisfying $h_1 = j^*((\pi^*)^{-1} f)|_{(Z/G)_{(H)}}$. Similarly, there exists $Y_2 \in \mathrm{Ham}((Z/G)_{(H)})$ which is Hamiltonian with respect to $h_2 = j^*((\pi^*)^{-1} f)|_{(Z/G)_{(H)}}$. So,

$$\begin{aligned}
i^*_{(H)}(\omega(X_f, X_g)) =& i^*_{(H)}\omega(X_f|_{Z_{(H)}}, X_g|_{Z_{(H)}}) && \text{by Proposition 2.39} \\
=& \pi^*_{(H)}\omega_{(H)}(X_f|_{Z_{(H)}}, X_g|_{Z_{(H)}}) \\
=& \pi^*_{(H)}(\omega_{(H)}(Y_1, Y_2)).
\end{aligned}$$

□

Poisson and Symplectic Structures.

Definition 2.42. A Poisson bracket on a differential structure $\mathcal{F}$ on a differential space X is a Lie bracket $\{,\}$ satisfying for any $f, g, h \in \mathcal{F}$:

$$\{f, gh\} = h\{f, g\} + g\{f, h\}.$$

Example 2.43. Define $\{,\}$ on (M, ω) by

$$\{f, g\} := \omega(X_f, X_g).$$

This is the standard Poisson structure on a symplectic manifold.

Theorem 2.44 (Lerman-Sjamaar)**.** *For each closed subgroup $H \leq G$ such that $Z_{(H)}$ is nonempty, the manifold $(Z/G)_{(H)}$ admits a symplectic form $\omega_{(H)} \in \Omega^2((Z/G)_{(H)})$ satisfying*

$$(\pi_{(H)})^*\omega_{(H)} = (i_{(H)})^*\omega.$$

Proof. See [14]. □

Since these manifolds $(Z/G)_{(H)}$ are symplectic, their rings of functions admit Poisson structures $\{\cdot,\cdot\}_{(H)}$ as in Example 2.43. In fact, we can define a Poisson bracket on all of Z/G as follows.

Definition 2.45. Let $f, g \in C^\infty(Z/G)$, and let $x \in (Z/G)_{(H)}$ for some $H \leq G$. Then define

$$\{f,g\}_{Z/G}(x) := \{f|_{(Z/G)_{(H)}}, g|_{(Z/G)_{(H)}}\}_{(H)}(x).$$

Proposition 2.46 (Lerman-Sjamaar)**.** *The above bracket defines a Poisson bracket on $C^\infty(Z/G)$.*

Proof. See [14]. □

We can also define a Poisson structure on $C^\infty(M/G)$:

Definition 2.47. Let $f, g \in C^\infty(M/G)$. Then define $\{f,g\}_{M/G} := (\pi^*)^{-1}\{\pi^*f, \pi^*g\}$, where $(\pi^*)^{-1}$ is the inverse of the isomorphism $\pi^* : C^\infty(M/G) \to C^\infty(M)^G$.

Proposition 2.48. *The above bracket defines a Poisson structure on $C^\infty(M/G)$.*

Proof. $C^\infty(M)^G$ is a *Poisson subalgebra* of $C^\infty(M)$ and π^* is an isomorphism between $C^\infty(M/G)$ and $C^\infty(M)^G$. The result follows. □

Definition 2.49. Let $(X, \mathcal{F}, \{,\}_X)$ and $(Y, \mathcal{G}, \{,\}_Y)$ be differential spaces equipped with Poisson structures. A smooth map $F : X \to Y$ is *Poisson* if for every $f, g \in \mathcal{G}$, $F^*(\{f,g\}_Y) = \{F^*f, F^*g\}_X$.

Proposition 2.50. *π and j are Poisson morphisms.*

Proof. By definition of the Poisson structure on $C^\infty(M/G)$, π is a Poisson map. As for j, fix $z \in Z_{(H)}$ and let $x = \pi_{(H)}(z)$. Let $f, g \in C^\infty(M/G)$. Then,

$$\begin{aligned} j^*\{f,g\}_{M/G}(x) =& \{f,g\}_{M/G}(j(x)) \\ =& \{\pi^*f, \pi^*g\}(i(z)) \\ =& \omega(X_{\pi^*f}, X_{\pi^*g})(i(z)) \\ =& i^*_{(H)}(\omega(X_{\pi^*f}, X_{\pi^*g}))(z) \\ =& \pi^*_{(H)}(\omega_{(H)}(Y_1, Y_2))(z) \qquad\qquad \text{by Lemma 2.41} \end{aligned}$$

where $Y_1 = X_{h_1}$ and $Y_2 = X_{h_2}$ with $h_1 = j^*((\pi^*)^{-1}\pi^*f)|_{(Z/G)_{(H)}} = j^*f|_{(Z/G)_{(H)}}$, and a similar formula for h_2 replacing f with g. Thus, we have

$$j^*\{f,g\}_{M/G}(x) = \{j^*f|_{(Z/G)_{(H)}}, j^*g|_{(Z/G)_{(H)}}\}_{(H)}(x) = \qquad \{j^*f, j^*g,\}Z/G(x).$$

This completes the proof. □

Before continuing, we summarise our setting so far. Diagramme 1 is in the category of smooth, stratified spaces, M, M/G and Z/G come equipped with Poisson structures on their rings of smooth functions, and π and j are Poisson morphisms with respect to these Poisson structures.

The de Rham Complex on a Symplectic Quotient. From the Principal Orbit Theorem 2.14, we know that there exists a closed subgroup $K \subseteq G$ such that $(Z/G)_{(K)}$ is open, connected and dense in Z/G. We call this stratum the *principal* stratum.

Definition 2.51. A *differential form* on Z/G is a differential k-form $\mu \in \Omega^k((Z/G)_{(K)})$ such that there exists $\tilde{\mu} \in \Omega^k(M)$ satisfying $\pi^*_{(K)}\mu = i^*_{(K)}\tilde{\mu}$. Denote the set of all k-forms on Z/G by $\Omega^k(Z/G)$.

Remark 2.52.

(1) $(\Omega(Z/G), d)$ forms a complex, where $\Omega^0(Z/G) = C^\infty(Z/G)$ and $\Omega(Z/G) := \bigoplus_{k=0}^\infty \Omega^k(Z/G)$.
(2) In the definition of $\mu \in \Omega^k(Z/G)$, we can choose $\tilde{\mu}$ to be G-invariant (just average over G). We will show later that in fact we can choose it to be basic on M.

Definition 2.53. A differential form $\eta \in \Omega(M)$ is Φ-*basic* if it is G-invariant and $i^*_{(K)}\eta \in \Omega(Z_{(K)})$ is horizontal. Hence $i^*_{(K)}\eta$ is basic on $Z_{(K)}$. Following Sjamaar's notation in [29], denote the set of all Φ-basic forms on M by $\Omega_\Phi(M)$, and the kernel of the natural map $\Omega_\Phi(M) \to \Omega(Z/G)$ by $I_\Phi(M)$; that is,

$$I_\Phi(M) := \{\eta \in \Omega(M)^G \mid i^*_{(K)}\eta = 0\}.$$

We thus have that

$$\Omega(Z/G) \cong \Omega_\Phi(M)/I_\Phi(M). \tag{3}$$

Lemma 2.54. *Let $\eta \in \Omega_\Phi(M)$. Then for all closed subgroups $H \leq G$ such that $Z_{(H)}$ is nonempty, $(i|_{Z_{(H)}})^*\eta$ is basic on $Z_{(H)}$.*

Proof. See [29]. □

Remark 2.55. In his paper [29], Sjamaar proves a Poincaré Lemma, a version of Stokes' Theorem and a de Rham theorem for the complex $(\Omega(Z/G), d)$.

We now come to the main theorem of the paper, which we now word in the language developed thus far.

Let G be a compact connected Lie group acting in a Hamiltonian fashion on connected symplectic manifolds (M, ω) and (M', ω'), each with proper momentum maps Φ and Φ', respectively. Let $Z := \Phi^{-1}(0)$ and $Z' := \Phi'^{-1}(0)$.

Main Theorem. *If F is a diffeomorphism from Z/G to Z'/G, then F induces an isomorphism of de Rham complexes $(\Omega(Z/G), d)$ and $(\Omega(Z'/G), d)$.*

To prove this theorem, it will be enough to show that $(\Omega(Z/G), d)$ depends only on the smooth structure $C^\infty(Z/G)$. The rest of this paper, except for the last section, is dedicated to this goal.

3. Vector Fields

As seen in the previous section, stratifications play an important role in the definition of differential forms on Z/G. The goal of this section is to replace strata with orbits of certain Lie algebras of vector fields. First, we will have to define vector fields on a subcartesian space and describe their properties.

The Zariski Tangent Bundle.

Definition 3.1. Given a point $x \in S$, a *derivation* of $C^\infty(S)$ at x is a linear map $v : C^\infty(S) \to \mathbb{R}$ that satisfies Leibniz' rule: for all $f, g \in C^\infty(S)$,

$$v(fg) = f(x)v(g) + g(x)v(f).$$

The set of all derivations of $C^\infty(S)$ at x form a vector space, called the *(Zariski) tangent space* of x, and is denoted T_xS. Define the *(Zariski) tangent bundle* TS to be the (disjoint) union

$$TS := \bigcup_{x \in S} T_xS.$$

Denote the canonical projection $TS \to S$ by τ.

TS is a subcartesian space with its differential structure generated by functions $f \circ \tau$ and df where $f \in C^\infty(S)$ and d is the differential operator $df(v) := v(f)$. The projection τ is smooth with respect to this differential structure. Given a chart $\varphi : U \to \tilde{U} \subseteq \mathbb{R}^n$ on S, $(\varphi \circ \tau, \varphi_*|_{\varphi \circ \tau}) : TS \to T\mathbb{R}^n \cong \mathbb{R}^{2n}$ is a fibrewise linear chart on TS. See [15] for more details. We will denote this chart as φ_* henceforth.

Definition 3.2. Let R and S be subcartesian spaces, and let $F : R \to S$ be a smooth map. Then there is an induced fibrewise linear smooth map $F_* : TR \to TS$ defined by

$$(F_*v)f = v(F^*f)$$

for all $v \in TR$ and $f \in C^\infty(S)$. F_* satisfies the following commutative diagramme.

$$\begin{tikzcd} TR \arrow[r, "F_*"] \arrow[d, "\tau"'] & TS \arrow[d, "\tau"] \\ R \arrow[r, "F"'] & S \end{tikzcd}$$

F_* is called the *pushforward* of F, and is sometimes denoted as dF or TF.

Proposition 3.3. *Let $x \in S$ and let $\varphi : U \to \tilde{U} \subseteq \mathbb{R}^n$ be a chart about x. Then, $\tilde{v} \in T_{\varphi(x)}\mathbb{R}^n$ is equal to φ_*v for some $v \in T_xS$ if and only if $\tilde{v}(\mathfrak{n}(\varphi(U))) = \{0\}$.*

Proof. See [15] $\square$

Definition 3.4. A *(global) derivation* of $C^\infty(S)$ is a linear map $X : C^\infty(S) \to C^\infty(S)$ that satisfies Leibniz' rule: for any $f, g \in C^\infty(S)$,

$$X(fg) = fX(g) + gX(f).$$

Denote the $C^\infty(S)$-module of all global derivations by $\mathrm{Der}C^\infty(S)$.

Proposition 3.5. *The set of derivations of $C^\infty(S)$ is a Lie algebra under the commutator bracket, and can be identified with the smooth sections of $\tau : TS \to S$.*

Proof. See [15]. □

Proposition 3.6. *Let $x \in S$, and let $\varphi : U \to \tilde{U} \subseteq \mathbb{R}^n$ be a chart about x. Let $\tilde{X} \in \mathrm{Der}C^\infty(\mathbb{R}^n)$. Then $\tilde{X}$ satisfies*

$$\varphi_*(X|_U) = \tilde{X}|_{\tilde{U}}$$

for some derivation $X \in \mathrm{Der}C^\infty(S)$ if and only if

$$\tilde{X}(\mathfrak{n}(\tilde{U})) \subseteq \mathfrak{n}(\tilde{U}).$$

Moreover, for any $X \in \mathrm{Der}C^\infty(S)$, there exist an open neighbourhood $V \subseteq U$ of x and $\tilde{X} \in \mathrm{Der}C^\infty(\mathbb{R}^n)$ satisfying $\varphi_*(X|_V) = \tilde{X}|_{\varphi(V)}$. We call $\tilde{X}$ a local extension *of X with respect to φ.*

Proof. See [15]. □

Definition 3.7. Let R and S be subcartesian spaces, and let f be a surjective smooth map between them. Then $f : R \to S$ is *locally trivial* if there exists a subcartesian space F satisfying: for every $x \in S$ there exist an open neighbourhood $U \subseteq S$ of x and a diffeomorphism $\psi : f^{-1}(U) \to U \times F$ such that the following diagramme commutes (pr_1 being the projection of the first component.)

$$\begin{tikzcd} f^{-1}(U) \arrow[rr, "\psi"] \arrow[dr, "f"'] & & U \times F \arrow[dl, "\mathrm{pr}_1"] \\ & U & \end{tikzcd}$$

Theorem 3.8. *There exists an open dense subset $U \subseteq S$ such that $\tau|_U : TS|_U \to U$ is locally trivial.*

Proof. See [15]. □

Remark 3.9. The kth exterior product of fibres of TS over S are also locally trivial over an open dense subset of S.

Vector Fields on Subcartesian Spaces.

Definition 3.10. Fix $X \in \mathrm{Der}C^\infty(S)$ and $x \in S$. A *maximal integral curve* $\exp(\cdot X)(x)$ of X through x is a smooth map from a maximal connected subset $I_x^X \subseteq \mathbb{R}$ containing 0 to S such that $\exp(0X)(x) = x$ and such that the following diagramme commutes.

$$\begin{tikzcd} TI_x^X \arrow[r, "\exp(\cdot X)(x)_*"'] & TS \\ I_x^X \arrow[u, "\frac{d}{dt}"] \arrow[r, "\exp(\cdot X)(x)"] & S \arrow[u, "X"'] \end{tikzcd}$$

In other words, for all $f \in C^\infty(S)$ and $t \in I_x^X$,

$$\frac{d}{dt}(f \circ \exp(tX)(x)) = (Xf)(\exp(tX)(x)).$$

We adopt the convention that the map $c : \{0\} \to S : 0 \mapsto c(0)$ is an integral curve of every global derivation of $C^\infty(S)$.

Theorem 3.11 (ODE Theorem for Subcartesian Spaces – Śniatycki)**.** *Let S be a subcartesian space, and let $X \in \mathrm{Der}C^\infty(S)$. Then, for any $x \in S$, there exists a unique maximal integral curve* $\exp(\cdot X)(x)$ *through x.*

Proof. See [30] and §4 Theorem 1 of [31]. $\square$

For each $X \in \mathrm{Der}C^\infty(S)$, there is an induced smooth map $\mathbb{R} \times S \supseteq A^X \to S$ where each fibre $A^X \cap (\mathbb{R} \times \{x\})$ is the domain I_x^X of the maximal integral curve $\exp(\cdot X)(x)$.

Proposition 3.12. *Let $\varphi : U \to \tilde{U} \subseteq \mathbb{R}^n$ be a chart on S, $X \in \mathrm{Der}C^\infty(S)$ and $\tilde{X} \in \mathrm{Der}C^\infty(\mathbb{R}^n)$ such that*

$$\varphi_*(X|_U) = \tilde{X}|_{\tilde{U}}.$$

Then for all $x \in S$ and $t \in I_x^X$ such that $\exp(tX)(x) \in U$*,*

$$\varphi(\exp(tX)(x)) = \exp(t\tilde{X})(\varphi(x)).$$

Proof. Denote by J the open subset of I_x^X such that for every $t \in J$, $\exp(tX)(x) \in U$. Define $\gamma : J \to \tilde{U} : t \mapsto \varphi(\exp(tX)(x))$. Then,

$$\begin{aligned} \frac{d}{dt}\Big|_{t=0}(\gamma(t)) =& \varphi_*(X|_x) \\ =& \tilde{X}|_{\varphi(x)}. \end{aligned}$$

Applying the ODE theorem, $\gamma(t) = \exp(t\tilde{X})(\varphi(x))$. $\square$

Definition 3.13. Let D be an open subset of $\mathbb{R}\times S$ containing $\{0\}\times S$ such that $D\cap(\mathbb{R}\times\{x\})$ is connected for each $x \in S$. A map $\phi : D \to S$ is a *local flow* if $\phi(0, x) = x$ for each $x \in S$ and $\phi(t, \phi(s, x)) = \phi(t + s, x)$ for all $x \in S$ and $s, t \in \mathbb{R}$ for which both sides are defined.

Remark 3.14. If S is a smooth manifold, then every derivation X admits a local flow $\exp(\cdot X)(\cdot)$ sending (t, x) to $\exp(tX)(x)$. This is not the case with subcartesian spaces, which motivates the following definition.

Definition 3.15. A *vector field* on S is a derivation X of $C^\infty(S)$ such that $\exp(\cdot X)(\cdot)$ is a local flow. Denote the set of all vector fields on S by $\mathrm{Vect}(S)$.

Remark 3.16. Given a vector field X on S, the domain A^X of the local flow $\exp(\cdot X)(\cdot)$ is open.

Proposition 3.17 (Śniatycki)**.** *Let S be a locally compact subcartesian space. A derivation X of $C^\infty(S)$ is a vector field if and only if the domain of each of its maximal integral curves is open.*

Proof. See §4 Proposition 3 of [31]. $\square$

Example 3.18. Consider the closed ray $[0, \infty) \subset \mathbb{R}$. The global derivation ∂_x is not a vector field on $[0, \infty)$. At any point, the maximal integral curve associated to ∂_x is not open. However, $x\partial_x$ is a vector field. The maximal integral curve at 0 is constant, and hence has domain $\mathbb{R}$. The maximal integral curves at all other points are also open.

Definition 3.19. Consider the subcartesian space Z/G in Diagramme 1. A vector field $X \in \mathrm{Vect}(Z/G)$ is called *Hamiltonian* if there exists $h \in C^\infty(Z/G)$ such that $X = \{h, \cdot\}_{Z/G}$. We will usually denote X by X_h, and the set of all Hamiltonian vector fields by $\mathrm{Ham}(Z/G)$, similar to the case for manifolds.

Lemma 3.20. *For any $h \in C^\infty(Z/G)$, the derivation $\{h, \cdot\}_{Z/G}$ is a Hamiltonian vector field.*

Proof. Lerman-Sjamaar prove the existence and uniqueness of maximal integral curves of these derivations, and that they preserve the connected components of the orbit-type strata (see [14]). Since these connected components are manifolds, the maximal integral curves have open domains. Hence, by Proposition 3.17, they are vector fields. □

Locally Complete Families of Vector Fields. We are interested in families of vector fields that yield a sufficiently "nice" partition of a subcartesian space. The condition needed to achieve this on these families is defined next. We then give examples.

Definition 3.21 (Śniatycki [31])**.** A family of vector fields $\mathcal{F} \subseteq \mathrm{Vect}(S)$ is *locally complete* if for every $X, Y \in \mathcal{F}$, every $x \in S$ and every $t \in \mathbb{R}$ such that $(\exp(tX)_* Y)|_x$ is well-defined, there exist an open neighbourhood U of x and a vector field $Z \in \mathcal{F}$ such that $\exp(tX)_* Y|_U = Z|_U$.

Example 3.22. Consider $S = \mathbb{R}^2$, and let $\mathcal{F} = \{\partial_x, x\partial_y\}$. This family is not locally complete, as one can check that $\exp(tx\partial_y)_*\partial_x = \partial_x + t\partial_y$ is not contained in $\mathcal{F}$ for any $t \neq 0$.

Proposition 3.23 (Śniatycki)**.** $\mathrm{Vect}(S)$ *is locally complete.*

Proof. See §4 Theorem 2 of [31]. □

Remark 3.24. The proof of the above proposition can be used to understand local completeness in terms of local flows. In particular, consider

$$\begin{aligned}\frac{d}{ds} f(\exp(tX)(\exp(sY)(x))) =& (\exp(tX)_*(Y|_{\exp(sY)(x)}))f \\ =& ((\exp(tX)_*Y)f)(\exp(tX)(\exp(sY)(x))).\end{aligned}$$

For fixed t, by Proposition 3.23, there exists an open neighbourhood U of x on which the local flow of $(\exp(tX)_*Y)|_U$ is equal to $s \mapsto \exp(tX)(\exp(sY)(y))$ for $y \in U$.

Definition 3.25. Let S be a subcartesian space that is also a stratified space. If the strata obtain their smooth structures as differential subspaces of S, then we refer to S as a *smooth stratified space*. We say that the stratification on a smooth stratified space S is *locally trivial* if for every $x \in S$ there exist

(1) an open neighbourhood $U \subseteq S$ of x,
(2) a smooth stratified space R with a distinguished point $p \in R$ such that $\{p\}$ is a stratum of R,
(3) and a stratified diffeomorphism $\varphi : U \to (P \cap U) \times R$ where P is the stratum of S containing x and for every $y \in (P \cap U)$, $\varphi(y) = (y, p)$.

Let $X \in \mathrm{Vect}(S)$. If for each stratum P of S, $X|_P$ is a smooth vector field on P, then we call X *stratified.*

Remark 3.26. Different terminology and definitions appear in the literature. For example, in [31], Śniatycki defines a smooth stratified space as a stratified topological space equipped with a special atlas of charts such that the strata obtain their smooth structures from the atlas. As a theorem, he proves that these are subcartesian spaces. Also, in the same article, a stratified vector field on a smooth stratified space S is not necessarily a smooth section, but a continuous section that is smooth on the strata. Instead, a strongly stratified vector field is an element of $\mathrm{Vect}(S)$ that restricts to a smooth section of each strata.

Example 3.27. In the setting of Diagramme 1, the orbit-type stratifications on all four spaces are locally trivial. (See [14].)

Lemma 3.28 (Śniatycki)**.** *Let S be a smooth stratified space with locally trivial stratification. Then the set of all stratified vector fields in* $\mathrm{Vect}(S)$ *form a locally complete family.*

Proof. See §6 Lemma 11 of [31] □

Returning to the situation in Diagramme 1, we have the following propositions.

Lemma 3.29. $\mathrm{Vect}(M)^G$ *is a locally complete Lie subalgebra of* $\mathrm{Vect}(M)$*.*

Proof. This proof is similar to the that of Lemma 3.57.

□

Definition 3.30. Identify $\mathfrak{g}$ with the left-invariant (under left multiplication) vector fields on G. Let $\rho : \mathfrak{g} \to \mathrm{Vect}(M)$ be the $\mathfrak{g}$-action induced by the G-action. Note that for all $x \in M$,

$$\check{T}_x^{\rho(\mathfrak{g})} M = T_x(G \cdot x).$$

Lemma 3.31. $\rho(\mathfrak{g})$ *is a locally complete Lie subalgebra of* $\mathrm{Vect}(M)$*.*

Proof. This proof is similar to that of Lemma 3.59. □

Recall that for a compact Lie group G, its Lie algebra decomposes as a direct sum of the derived Lie subalgebra and the centre of $\mathfrak{g}$:

$$\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}] \oplus \mathfrak{z}(\mathfrak{g}).$$

Corollary 3.32. $\rho([\mathfrak{g}, \mathfrak{g}])$ *and* $\rho(\mathfrak{z}(\mathfrak{g}))$ *are locally complete.*

Proof. Since $[\mathfrak{g}, \mathfrak{g}]$ and $\mathfrak{z}(\mathfrak{g})$ are themselves Lie algebras, this corollary is immediate from the above lemma. □

Definition 3.33. Define $\mathcal{A}$ to be the smallest Lie subalgebra of $\mathrm{Vect}(M)$ containing $\mathrm{Vect}(M)^G$ and $\rho(\mathfrak{g})$.

Remark 3.34. Note that $\mathcal{A}$, $\mathrm{Vect}(M)^G$ and $\rho(\mathfrak{g})$ are not necessarily closed under multiplication by functions in $C^\infty(M)$, but they are closed under multiplication by G-invariant smooth functions.

Lemma 3.35. $\mathcal{A}$ *is locally complete, and it is a direct sum of Lie algebras:*

$$\mathcal{A} = \rho([\mathfrak{g}, \mathfrak{g}]) \oplus \mathrm{Vect}(M)^G.$$

Proof. Let $\xi \in \mathfrak{g}$ and $X \in \mathrm{Vect}(M)^G$. Then,

$$[\xi_M, X] = \lim_{t \to 0} \frac{\exp(t\xi_M)_*(X|_{\exp(-t\xi_M)(x)}) - X|_x}{t} = 0$$

since $\exp(t\xi_M)_*(X|_{\exp(-t\xi_M)(x)}) = X|_x$ by left-invariance. Thus,

$$\exp(t\xi_M) \circ \exp(sX) = \exp(sX) \circ \exp(t\xi_M). \tag{4}$$

Now, let $\xi \in \mathfrak{g}$ and assume for all $g \in G$ and $x \in M$, we have

$$g_*(\xi_M|_x) = \xi_M|_{g \cdot x};$$

that is, ξ_M is left-invariant. Then,

$$\frac{d}{dt}\Big|_{t=0}(g \cdot \exp(t\xi_M)(x)) = \frac{d}{dt}\Big|_{t=0} \exp(t\xi_M)(g \cdot x).$$

The uniqueness property of exp implies that

$$g \cdot \exp(t\xi_M)(x) = \exp(t\xi_M)(g \cdot x).$$

Hence $(g \exp(t\xi)) \cdot x = (\exp(t\xi)g) \cdot x$. Since this is true for all $g \in G$, $\exp(t\xi)$ must be in the centre of G, and hence $\xi \in \mathfrak{z}(\mathfrak{g})$. Thus,

$$\rho(\mathfrak{g}) \cap \mathrm{Vect}(M)^G = \rho(\mathfrak{z}(\mathfrak{g})).$$

Since ρ is a Lie algebra homomorphism, from Equation 4: $\rho(\mathfrak{g}) = \rho([\mathfrak{g}, \mathfrak{g}]) \oplus \rho(\mathfrak{z}(\mathfrak{g}))$, and we obtain the direct sum structure of $\mathcal{A}$.

To show local completeness, it suffices to show that for any $\xi \in \mathfrak{g}$ and $X \in \mathrm{Vect}(M)^G$, $\exp(t\xi_M)_* X \in \mathcal{A}$ and $\exp(tX)_* \xi_M \in \mathcal{A}$. The former is immediate since X is left-invariant. The latter follows from Equation 4:

$$\begin{aligned}
\exp(tX)_*(\xi_M|_x) =& \frac{d}{ds}\Big|_{s=0} \exp(tX)(\exp(s\xi_M)(x)) \\
=& \frac{d}{ds}\Big|_{s=0} \exp(s\xi_M)(\exp(tX)(x)) \\
=& \xi_M|_{\exp(tX)(x)}.
\end{aligned}$$

□

Lemma 3.36. $\mathrm{Ham}(Z/G)$ *is a locally complete family in* $\mathrm{Vect}(Z/G)$.

Proof. See §7 Proposition 4 in [31]. □

The Orbital Tangent Bundle.

Definition 3.37. Let $\mathcal{F}$ be a family of vector fields on S. For $x \in S$, denote by $\check{T}^{\mathcal{F}}_x S$ the linear subspace of $T_x S$ spanned by all vectors $v \in T_x S$ such that there exists a vector field $X \in \mathcal{F}$ with $v = X|_x$. If $\mathcal{F} = \mathrm{Vect}(S)$, then we will denote this space by $\check{T}_x S$. We will call $\check{T}^{\mathcal{F}}_x S$ the *orbital tangent space* of S at x with respect to $\mathcal{F}$. Let $\check{T}^{\mathcal{F}} S$ be the (disjoint) union

$$\check{T}^{\mathcal{F}} S := \bigcup_{x \in S} \check{T}^{\mathcal{F}}_x S.$$

We will call $\check{T}^{\mathcal{F}} S$ the *orbital tangent bundle* with respect to $\mathcal{F}$. It is a differential subspace of TS. Denote by $\check{\tau}_{\mathcal{F}}$ the restriction of $\tau : TS \to S$ to $\check{T}^{\mathcal{F}} S$ and by $\delta_{\mathcal{F}}(x)$ the dimension $\dim(\check{T}^{\mathcal{F}}_x S)$.

Remark 3.38. Since $\check{T}^{\mathcal{F}} S$ is a differential subspace of TS, a chart $\varphi : U \to \tilde{U} \subseteq \mathbb{R}^n$ on S induces a chart $(\varphi \circ \check{\tau}_{\mathcal{F}}, \varphi_*|_{\varphi \circ \check{\tau}_{\mathcal{F}}})$ on $\check{T}^{\mathcal{F}} S$, which we shall denote simply as φ_*. This is just a restriction of the corresponding chart on TS. It makes the following diagramme commute.

$$\begin{tikzcd}
\check{T}^{\mathcal{F}} S|_U \arrow[r, "\varphi_*"] \arrow[d, "\check{\tau}_{\mathcal{F}}"'] & T\mathbb{R}^n \arrow[d, "\tau"] \\
U \arrow[r, "\varphi"'] & \mathbb{R}^n
\end{tikzcd}$$

This extends to (fibred) exterior powers of $\check{T}^{\mathcal{F}}S$ in the natural way; *i.e.* to

$$\bigwedge_S^k \check{T}^{\mathcal{F}}S := \bigcup_{x\in S} \bigwedge^k \check{T}_x^{\mathcal{F}}S.$$

Lemma 3.39. *The map $\delta_{\mathcal{F}} : S \to \mathbb{Z}$ is lower semicontinuous.*

Proof. Define $S_i := \{x \in S \mid \delta_{\mathcal{F}}(x) \geq i\}$. The goal is to show that S_i is open for each i. Let $y \in S_i$. Then there exist $Y_1, ..., Y_k \in \mathcal{F}$, where $k \geq i$, such that $\{Y_1|_y, ..., Y_k|_y\}$ is a basis for $\check{T}_y^{\mathcal{F}}S$. Linear independence is an open condition, and so there exists an open neighbourhood U of y such that $\{Y_1|_z, ..., Y_k|_z\}$ is linear independent for all $z \in U$. Hence, $\check{T}_z^{\mathcal{F}}S$ contains the span of $\{Y_1|_z, ..., Y_k|_z\}$ as a linear subspace for each $z \in U$. Thus, $\delta_{\mathcal{F}}(z) \geq k \geq i$. Thus, $U \subseteq S_i$. □

Proposition 3.40. *There exists an open dense subset $U \subseteq S$ such that $\check{\tau}_{\mathcal{F}}|_U : \check{T}^{\mathcal{F}}S|_U \to U$ is locally trivial.*

Proof. We will show that for any point $x \in S$ and any open set U containing x, there is a point $z \in U$ and an open neighbourhood $V \subseteq U$ of z so that $\check{\tau}_{\mathcal{F}}^{-1}(V) \cong V \times F$ for some vector space F.

Fix $x \in S$. Define S_i as in the proof of Lemma 3.39. Define

$$m := \inf_{V \ni x}\{\sup\{k \mid S_k \cap V \neq \emptyset\}\}$$

where V runs through all open neighbourhoods of x. There exists an open neighbourhood W of x such that $\sup_{z\in W}\{\delta_{\mathcal{F}}(z)\} = m$. Now fix $z \in W$ such that $\delta_{\mathcal{F}}(z) = m$. Then, there are vector fields $Y_1, ..., Y_m \in \mathcal{F}$ such that $\{Y_1|_z, ..., Y_m|_z\}$ spans $\check{T}_z^{\mathcal{F}}S$. Since linear independence is an open condition and m is maximal, there is an open neighbourhood $V \subseteq W$ of z such that $\{Y_1|_y, ..., Y_m|_y\}$ spans $\check{T}_y^{\mathcal{F}}S$ for all $y \in V$. Hence, $\check{T}^{\mathcal{F}}S$ is locally trivial over V.

Now, let U be any open subset containing x. We claim that there exists some $z \in W \cap U$ such that $\delta_{\mathcal{F}}(z) = m$. Assume otherwise. If $\sup_{z\in W\cap U}(\delta_{\mathcal{F}}(z)) > m$, then this contradicts the definition of W. If $\sup_{z\in W\cap U}\{\delta_{\mathcal{F}}(z)\} < m$, then this contradicts the definition of m. Now, choose an open neighbourhood $V \subseteq W \cap U$ of z as above, and the result follows. □

Corollary 3.41. *Let $\mathcal{F}$ be a locally complete family of vector fields, and let $U \subseteq S$ be an open dense subset on which $\check{T}^{\mathcal{F}}S$ is locally trivial. Then, $\check{\tau}_{\mathcal{F}}^{-1}(U)$ is open and dense in $\check{T}^{\mathcal{F}}S$.*

Proof. By continuity, $\check{\tau}_{\mathcal{F}}(U)$ is open. Let $x \in S \setminus U$, and let $Y_1, ..., Y_k \in \mathcal{F}$ such that $\{Y_1|_x, ..., Y_k|_x\}$ forms a basis of $\check{T}_x^{\mathcal{F}}S$. Since linear independence is an open condition, there is an open neighbourhood V of x on which $\{Y_1|_y, ..., Y_k|_y\}$ is linear independent for all $y \in V$, and their span is a subset of $\check{T}_y^{\mathcal{F}}S$. Hence, $\check{T}_x^{\mathcal{F}}S \subseteq \overline{\check{\tau}_{\mathcal{F}}^{-1}(U)}$. □

Remark 3.42. The above corollary extends to exterior powers of the fibres of $\check{T}^{\mathcal{F}}S$; that is, there exists an open dense subset $U \subseteq S$ on which $\bigwedge_S^k \check{T}^{\mathcal{F}}S\Big|_U \to U$ is locally trivial.

Proposition 3.43. *Let S be a locally compact subcartesian space. Then there exists an open dense subset $U \subseteq S$ such that for each $x \in U$,*

$$\check{T}_xS = T_xS$$

.

Proof. By Theorem 3.8 and Proposition 3.40, there exists an open dense subset $U \subseteq S$ on which TS and $\check{T}S$ are locally trivial. Let $x \in U$, and let $\varphi : V \to \tilde{V} \subseteq \mathbb{R}^n$ be a chart about x where $V \subseteq U$ and $n = \dim(T_xS)$ (see [15]). Then the derivations $\partial_1, ..., \partial_n$ on V arising from coordinates on $\mathbb{R}^n$ give a local trivialisation of TV (again, see [15]). Let W_1 and W_2 be open neighbourhoods of x satisfying $\overline{W_1} \subset W_2 \subset \overline{W_2} \subset V$. Let $b : S \to \mathbb{R}$ be a smooth bump function that is equal to 1 on W_1 and 0 outside of W_2. Then $b\partial_1, ..., b\partial_n$ extend to derivations on all of S, and we claim that they are vector fields.

Now, for $i = 1, ..., n$, shrinking V if necessary, there exist $\tilde{X}_1, ..., \tilde{X}_n \in \mathrm{Der}C^\infty(\mathbb{R}^n)$ satisfying $\varphi_*(b\partial_i) = \tilde{X}_i|_{\tilde{V}}$. Each $\tilde{X}_i$ gives rise to a local flow $\exp(\cdot\tilde{X}_i)(\cdot)$, such that for each $y \in \tilde{V}$, $\exp(\cdot\tilde{X}_i)(\varphi(y))$ has an open domain. By Proposition 3.12, $\exp(\cdot\tilde{X}_i)(\varphi(y)) = \varphi(\exp(tb\partial_i)(y))$ for all $t \in I^{b\partial_i}_y$ for which the integral curve lies in V. But since b is supported in V, the entire curve $\exp(\cdot b\partial_i)(y)$ is in V. Hence, $\exp(t\tilde{X}_i)(\varphi(y)) \in \tilde{V}$ for all $t \in I^{\tilde{X}_i}_{\varphi(y)}$. Since $\tilde{X}_i$ is a vector field on $\mathbb{R}^n$, $I^{\tilde{X}_i}_{\varphi(y)}$ is open, and consequently so is $I^{b\partial_i}_y$. Thus, by Proposition 3.17 $b\partial_i$ is a vector field on V, and since it has been extended as 0 to the rest of S, it is a vector field on S. Finally, since $(b\partial_i)|_{W_1} = \partial_i|_{W_1}$ for each i, we see that $\check{T}_yS = T_yS$ for all $y \in W_1$. □

Orbits of Families of Vector Fields.

Definition 3.44. Let $\mathcal{F}$ be a family of vector fields. The *orbit* of $\mathcal{F}$ through a point x, denoted $O^{\mathcal{F}}_x$ or just O_x if $\mathcal{F} = \mathrm{Vect}(S)$, is the set of all points $y \in S$ such that there exist vector fields $X_1, ..., X_k \in \mathcal{F}$ and real numbers $t_1, ..., t_k \in \mathbb{R}$ satisfying

$$y = \exp(t_1X_1) \circ ... \circ \exp(t_kX_k)(x).$$

Denote by $\mathcal{O}_{\mathcal{F}}$, or just $\mathcal{O}$ if $\mathcal{F} = \mathrm{Vect}(S)$, the set of all orbits $\{O^{\mathcal{F}}_x \mid x \in S\}$. Note that $\mathcal{O}_{\mathcal{F}}$ induces a partition of S into connected differential subspaces.

Given a family of vector fields $\mathcal{F}$ on S, there exists a natural topology on the orbits that in general is finer than the subspace topology. We define this topology here using similar notation as found in [31] and [34]. Let $X_1, ..., X_k \in \mathcal{F}$. Let $\xi := (X_1, ..., X_k)$ and $T = (t_1, ..., t_k)$, and define $\xi_T(x) := \exp(t_kX_k) \circ ... \circ \exp(t_1X_1)(x)$. $\xi_T(x)$ is well-defined for all (T, x) in an open neighbourhood $U(\xi)$ of $(0, x) \in \mathbb{R}^k \times S$. Define $U_x(\xi)$ to be the set of all $T \in \mathbb{R}^k$ such that $\xi_T(x)$ is well-defined; that is, $U_x(\xi) = U(\xi) \cap (\mathbb{R}^k \times \{x\})$. Let $i : O^{\mathcal{F}}_x \hookrightarrow S$ be the inclusion map. Fix $y \in i(O^{\mathcal{F}}_x)$ and let $\varphi : V \to \tilde{V} \subseteq \mathbb{R}^n$ be a chart of S about y. We give $W := i^{-1}(V \cap i(O^{\mathcal{F}}_x))$ the finest topology such that for each ξ and $y \in i(W)$ the map

$$\rho_{\xi,y} : U_y(\xi) \to \mathbb{R}^n : T \mapsto \varphi \circ \xi_T(y)$$

is continuous. This extends to a topology $\mathcal{T}$ on all of $O^{\mathcal{F}}_x$, which matches on overlaps (see [31]).

Example 3.45. Let $S = \mathbb{R}^2/\mathbb{Z}^2$ and let $\pi : \mathbb{R}^2 \to S$ be the quotient map. Consider the one-element family $\{X\}$ where $X = \pi_*(\partial_1 + \sqrt{2}\partial_2)$. Then for any $x \in S$, $\exp(tX)(x)$ has domain $\mathbb{R}$, and the orbit is dense in S. $\mathcal{T}$ in this case is such that $O^{\{X\}}_x$ is diffeomorphic to $\mathbb{R}$. This is strictly finer than the subspace topology on the orbit.

Theorem 3.46. *Let S be a subcartesian space. Then for any locally complete family of vector fields $\mathcal{F}$, $\mathcal{O}_{\mathcal{F}}$ induces a partition of S into orbits $O_x^{\mathcal{F}}$, each of which when equipped with the topology $\mathcal{T}$ described above has a smooth manifold structure. The inclusion $i : O_x^{\mathcal{F}} \hookrightarrow S$ is smooth, and $i_* : TO_x^{\mathcal{F}} \to TS$ is a fibrewise linear isomorphism onto $\check{T}^{\mathcal{F}}S|_{O_x^{\mathcal{F}}}$.*

Proof. See §5 Theorem 3 of [31]. □

Example 3.47. In Example 3.22 the tangent space $\dim(\check{T}^{\mathcal{F}}_{(0,y)}\mathbb{R}^2) = 1$ for all y, whereas $\check{T}^{\mathcal{F}}_{(x,y)}\mathbb{R}^2 = T_{(x,y)}\mathbb{R}^2$ for $x \neq 0$. But there is only one orbit: all of $\mathbb{R}^2$. So the family of vector fields $\{\partial_x, x\partial_y\}$ does not satisfy the conclusion of Theorem 3.46.

Theorem 3.48. *[Śniatycki] Orbits of any family of vector fields $\mathcal{F}$ are contained within orbits of* $\operatorname{Vect}(S)$.

Proof. See §5 Theorem 4 of [31]. □

Theorem 3.49. *Let S be a smooth stratified space whose stratification is locally trivial. Then the orbits on S induced by* $\operatorname{Vect}(S)$ *form a smooth stratification.*

Proof. See §6 Theorem 8 of [31]. □

Theorem 3.50 (Śniatycki)**.** *The strata of the orbit-type stratification on M/G are precisely the orbits in $\mathcal{O}$ induced by* $\operatorname{Vect}(M/G)$.

Proof. The proof can be found in [31]. The idea is the following. By Theorem 2.16 the orbit-type stratification on M/G is minimal. The family of stratified vector fields of this stratification is locally complete by Lemma 3.28 and its orbits are the strata. By Theorem 3.48, these strata lie in orbits of $\operatorname{Vect}(M/G)$. But, the set of orbits $\mathcal{O}$ induced by $\operatorname{Vect}(M/G)$ themselves form a stratification of M/G by Theorem 3.49. So by minimality, we must have that these two stratifications are equal. □

Lemma 3.51. *Orbits of* $\operatorname{Ham}(Z/G)$ *are contained in orbit-type strata of Z/G.*

Proof. Lerman and Sjamaar showed in [14] that the maximal integral curves of any Hamiltonian vector field on Z/G is confined to a symplectic stratum. The result follows. □

Lie Algebras of Vector Fields. Our goal for this subsection is to establish that for a locally compact subcartesian space S, $\operatorname{Vect}(S)$ is a Lie algebra under the commutator bracket.

For a subset $A \subseteq S$ we shall denote by $\mathfrak{n}(A)$ the set of functions $\{f \in C^\infty(S) \mid f|_A = 0\}$.

Proposition 3.52. *Let S be a subcartesian space and $\mathcal{F}$ a locally complete family of vector fields. Let $x \in S$ and $v \in T_xS$. Then, $v \in \check{T}^{\mathcal{F}}_x S$ if and only if for every open neighbourhood $U \subseteq O_x^{\mathcal{F}}$ of $i^{-1}(x)$, where i is the inclusion of $O_x^{\mathcal{F}}$ into S, we have $v(\mathfrak{n}(i(U))) = \{0\}$.*

Proof. Let $v \in \check{T}^{\mathcal{F}}_x S$. Then by Theorem 3.46 $v = i_*w$ for some $w \in TO_x^{\mathcal{F}}$. For any open neighbourhood U of $i^{-1}(x)$ and for any $f \in \mathfrak{n}(i(U))$,

$$vf = w(i^*f) = 0.$$

Conversely, let $v \in T_xS$ and let $\varphi : V \to \tilde{V} \subseteq \mathbb{R}^n$ be a chart about x. Then, $\varphi(V \cap O_x^{\mathcal{F}})$ is a differential subspace of $\mathbb{R}^n$, and in fact since $\varphi \circ i|_{i^{-1}(V)}$ is smooth with $d(\varphi \circ i|_{i^{-1}(V)})$ one-to-one (by Theorem 3.46), we have that $\varphi \circ i|_{i^{-1}(V)}$ is an immersion. Hence by the rank

theorem there exists an open neighbourhood $U \subseteq i^{-1}(V)$ of $i^{-1}(x)$ such that $\tilde{U} := \varphi \circ i(U)$ is an embedded submanifold of $\mathbb{R}^n$.

Now, v has a unique extension to a vector $\tilde{v} = \varphi_* v \in T_x\mathbb{R}^n$. Suppose $vf = 0$ for all $f \in \mathfrak{n}(i(U))$. Then for each such f, by Proposition 3.3, $\tilde{v}\tilde{f} = 0$ for any local representative $\tilde{f}$ of f. But then, also by Proposition 3.3, we have that $\tilde{v}$ is the unique local extension of a vector $\tilde{w} \in T_{\varphi(x)}\tilde{U}$ since $\tilde{f}|_{\tilde{U}} = 0$. Since $\tilde{U}$ is an embedded submanifold, there exists a unique $w \in T_{i^{-1}(x)}U$ such that $(\varphi \circ i)_* w = \tilde{w}$, which we identify with $\tilde{v}$. By Theorem 3.46 and uniqueness, $i_* w = v$. Thus, $v \in \check{T}^{\mathcal{F}}_x S$.

Since any open neighbourhood W of $i^{-1}(x)$ contains a smaller open neighbourhood $U \subseteq i^{-1}(V) \cap W$ in which $\varphi \circ i(U)$ is an embedded submanifold of $\mathbb{R}^n$, and also $\mathfrak{n}(i(W)) \subseteq \mathfrak{n}(i(U))$, we can apply the above argument, obtaining our result. $\square$

Proposition 3.53. *Let S be a locally compact subcartesian space. A derivation $X \in \mathrm{Der}C^\infty(S)$ is a vector field if and only if for every $x \in S$, every open neighbourhood U of $i^{-1}(x)$ and every $f \in \mathfrak{n}(i(U))$,*

$$X(\mathfrak{n}(i(U))) \subseteq \mathfrak{n}(i(U)).$$

Proof. Let X be a vector field. Then for any $x \in S$ and any open neighbourhood U of $i^{-1}(x)$, $X|_{i(U)}$ is a vector field on $i(U)$. By Proposition 3.52 for any $f \in \mathfrak{n}(i(U))$,

$$(Xf)|_{i(U)} = 0.$$

Conversely, let X be a derivation of $C^\infty(S)$ satisfying the property that for any open neighbourhood U of $i^{-1}(x)$, $X(\mathfrak{n}(i(U))) \subseteq \mathfrak{n}(i(U))$ for all orbits O_x with inclusion $i : O_x \to S$. By Proposition 3.17, it is enough to show that each maximal integral curve of X has an open domain.

Assume otherwise: there exists a maximal integral curve $\exp(tX)(x)$ through a point $x \in S$ with a closed or half-closed domain I^X_x. If $X|_x = 0$, then $\exp(tX)(x)$ is a constant map, and its maximal integral curve has $\mathbb{R}$ as its domain, which is open. So assume $X|_x \neq 0$. Let $a \in I^X_x$ be an endpoint of I^X_x and let $y := \exp(aX)(x)$. Then for any open neighbourhood $U \subseteq O_y$ of $i^{-1}(y)$,

$$(Xf)|_{i(U)} = 0$$

for all $f \in \mathfrak{n}(i(U))$. In particular, $X|_z f = 0$ for all $f \in \mathfrak{n}(i(U))$ and all $z \in i(U)$. By Proposition 3.52, $X|_z \in \check{T}_z S$ for all $z \in i(U)$. Note that since $X|_x \neq 0$, we have that $X|_y \neq 0$, and so there exists an open neighbourhood $V \subseteq i(U)$ of y such that $X|_z \neq 0$ for all $z \in V$.

Thus, since $X|_V$ is a smooth section of $TV \subseteq TS$, by Theorem 3.46 we have constructed a vector field $Y \in \mathrm{Vect}(V)$ such that $Y|_z = X|_z$. But note that by Proposition 3.12 these integral curves locally are restrictions of integral curves in $\mathbb{R}^n$, and so we can apply the ODE theorem, and obtain that since $X|_V = Y$, we have $\exp(tX)(y) = \exp(tY)(y)$ for t in some domain I_y. But, shrinking V if necessary so that it is an embedded submanifold of S (which exists by the rank theorem), since Y is a vector field on the manifold V, I_y is open and contains 0, whereas since $\exp(tX)(y) = \exp((t + a)(X))(x)$, by assumption I_y has $0 \in I_y$ as

an endpoint. This is a contradiction. Thus, I_x^X does not contain any endpoints, and hence is open. $\square$

Corollary 3.54. *Let S be a locally compact subcartesian space. Then $\mathrm{Vect}(S)$ is a Lie subalgebra of $\mathrm{Der}C^\infty(S)$ and is a $C^\infty(S)$-module.*

Proof. Let $x \in S$, $X, Y \in \mathrm{Vect}(S)$, $U \subseteq O_x$ any open neighbourhood of $i^{-1}(x)$ and $f \in \mathfrak{n}(i(U))$ and $g \in C^\infty(S)$. Applying Proposition 3.53, we have $(X+Y)(f) = X(f)+Y(f) = 0$, $gX(f) = 0$ and $[X,Y](f) = X(Yf) - Y(Xf) = 0$. Thus, $X+Y$, gX and $[X,Y]$ are vector fields. $\square$

Remark 3.55. By the above corollary, for any $x \in S$ and any $v \in \check{T}_xS$, there is a vector field X such that $X|_x = v$. In other words, we did not need to take the linear span in the definition of $\check{T}_xS$.

We again return to the situation in Diagramme 1. We have shown that $\mathrm{Vect}(Z)$ is a Lie algebra. Denote by $\mathrm{Vect}(Z)^G$ the Lie subalgebra of G-invariant vector fields on Z.

Lemma 3.56. *Let $X \in \mathrm{Vect}(Z)^G$ and let $x \in Z \subset M$. Then there exist a G-invariant open neighbourhood $U \subseteq M$ of x and $\tilde{X} \in \mathrm{Vect}(M)^G$ such that*

$$X|_{U\cap Z} = \tilde{X}|_{U\cap Z}.$$

Proof. There exist an open neighbourhood $V \subseteq M$ of x and $\tilde{X}_0 \in \mathrm{Vect}(M)$ such that $\tilde{X}_0|_{V\cap Z} = X|_{V\cap Z}$. Let $g_0 = e \in G$ and let g_i be elements of G for $i = 1, ..., k$ such that $G\cdot x \subseteq M$ is covered by open sets $g_i \cdot V$. Let $\{\zeta_i\}$ be a partition of unity subordinate to this cover, and define

$$\tilde{X} := \sum_{i=0}^{k} \zeta_i g_{i*}\tilde{X}_0.$$

Then, letting $W := \bigcup_{i=0}^k g_i \cdot V$, we have that for any $y \in W \cap Z$

$$\begin{aligned}\tilde{X}|_y =& \sum_{i=0}^{k} \zeta_i(y) g_{i*}(\tilde{X}_0|_{g_i^{-1}\cdot y}) \\ =& \sum_{i=0}^{k} \zeta_i(y) g_{i*}(X|_{g_i^{-1}\cdot y}) \\ =& \sum_{i=0}^{k} \zeta_i(y) X|_y \\ =& X|_y.\end{aligned}$$

Thus, $\tilde{X} \in \mathrm{Vect}(M)$ is a local extension of X on $W \cap Z$. Averaging $\tilde{X}$ (that is, let $\bar{X}|_y := \int_G (g_*\tilde{X})|_y dg$, which is a smooth G-invariant vector field) and letting U be a G-invariant open neighbourhood of $G\cdot x$ contained in W, we are done. $\square$

Lemma 3.57. $\mathrm{Vect}(Z)^G$ *is a locally complete Lie subalgebra of* $\mathrm{Vect}(Z)$*.*

Proof. Since diffeomorphisms commute with the commutator bracket, we have that $\mathrm{Vect}(Z)^G$ is a Lie subalgebra of $\mathrm{Vect}(Z)$. For any two left-invariant vector fields X and Y, we have for all $g \in G$ and $x \in Z$

$$g \cdot \exp(tX)(\exp(sY)(x)) = \exp(tX)(\exp(sY)(g \cdot x))$$

for s, t such that the composition of the curves is defined. Thus $\exp(tX)_*Y$ is locally defined about G-orbits. Since $\mathrm{Vect}(Z)$ is locally complete, for any $x \in Z$ there exist a vector field Ξ on Z and an open neighbourhood U of x such that $\exp(tX)_*Y$ is defined on U and $(\exp(tX)_*Y)|_U = \Xi|_U$. Since $\exp(tX)_*Y$ is left-invariant about x, we can choose U to be a G-invariant open neighbourhood. Let $V \subset U$ be a G-invariant open neighbourhood of x such that $\overline{V} \subset U$. Let $b : M \to \mathbb{R}$ be a G-invariant smooth bump function with support in U and $b|_V = 1$. Then, $b\Xi \in \mathrm{Vect}(Z)^G$ extends $(\exp(tX)_*Y)|_V$ to a left-invariant vector field on Z. □

Definition 3.58. Let $\rho_Z : \mathfrak{g} \to \mathrm{Der} C^\infty(Z)$ be the $\mathfrak{g}$-action induced by the action of G on Z. Note that by Proposition 3.17, $\rho_Z(\mathfrak{g}) \subseteq \mathrm{Vect}(Z)$ (in fact, for any $\xi \in \mathfrak{g}$, $\xi_Z := \rho_Z(\xi)$ is just the restriction of ξ_M to Z).

Lemma 3.59. *$\rho_Z(\mathfrak{g})$ is a locally complete Lie subalgebra of* $\mathrm{Vect}(Z)$.

Proof. Let $\xi, \zeta \in \mathfrak{g}$, and let $\xi_Z = \rho_Z(\xi)$ and $\zeta_Z = \rho_Z(\zeta)$. Then, $\exp(t\xi_Z)_*\zeta_Z = (\mathrm{Ad}_{\exp(t\xi)}\zeta)_Z$. Thus $\rho_Z(\mathfrak{g})$ is locally complete, and since ρ_Z is a Lie algebra homomorphism, its image is a Lie algebra. □

Corollary 3.60. *$\rho_Z([\mathfrak{g}, \mathfrak{g}])$ and $\rho_Z(\mathfrak{z}(\mathfrak{g}))$ are both locally complete Lie subalgebras of* $\mathrm{Vect}(Z)$.

Proof. This is immediate from the above lemma. □

Definition 3.61. Define $\mathcal{A}_Z$ to be the smallest Lie subalgebra of $\mathrm{Vect}(Z)$ that contains both $\rho_Z(\mathfrak{g})$ and $\mathrm{Vect}(Z)^G$.

Lemma 3.62. *$\mathcal{A}_Z$ is locally complete and is equal to the direct sum of Lie subalgebras*

$$\mathcal{A}_Z = \rho_Z([\mathfrak{g}, \mathfrak{g}]) \oplus \mathrm{Vect}(Z)^G.$$

Proof. By Lemma 3.56, for any $X \in \mathrm{Vect}(Z)^G$ and for any $x \in Z$, there exist a G-invariant open neighbourhood $U \subseteq M$ of x and $\tilde{X} \in \mathrm{Vect}(M)^G$ such that

$$X|_{U \cap Z} = \tilde{X}|_{U \cap Z}.$$

Hence,

$$[\xi_Z, X]|_{U \cap Z} = [\xi_M, \tilde{X}]|_{U \cap Z} = 0$$

by Lemma 3.35. Thus, applying Proposition 3.12 and Equation 4 in the proof of Lemma 3.35, we have that

$$\exp(t\xi_Z) \circ \exp(sX) = \exp(sX) \circ \exp(t\xi_Z). \tag{5}$$

Now, let $\xi \in \mathfrak{g}$ and assume for all $g \in G$ and $x \in Z$, we have

$$g_*(\xi_Z|_x) = \xi_Z|_{g \cdot x}.$$

Then,

$$\frac{d}{dt}\Big|_{t=0} (g \cdot \exp(t\xi_Z)(x)) = \frac{d}{dt}\Big|_{t=0} \exp(t\xi_Z)(g \cdot x).$$

The uniqueness property of exp implies that

$$g \cdot \exp(t\xi_Z)(x) = \exp(t\xi_Z)(g \cdot x).$$

Hence $(g \exp(t\xi)) \cdot x = (\exp(t\xi)g) \cdot x$. Since this is true for all $g \in G$, $\exp(t\xi)$ must be in the centre of G, and hence $\xi \in \mathfrak{z}(\mathfrak{g})$. Thus,

$$\rho_Z(\mathfrak{g}) \cap \mathrm{Vect}(Z)^G = \rho_Z(\mathfrak{z}(\mathfrak{g})).$$

Since ρ_Z is a Lie algebra homomorphism, from Equation 5: $\rho_Z(\mathfrak{g}) = \rho_Z([\mathfrak{g},\mathfrak{g}]) \oplus \rho_Z(\mathfrak{z}(\mathfrak{g}))$, and we obtain the direct sum structure of $\mathcal{A}_Z$.

To show local completeness, it suffices to show that for any $\xi \in \mathfrak{g}$ and $X \in \mathrm{Vect}(Z)^G$, $\exp(t\xi_Z)_* X \in \mathcal{A}_Z$ and $\exp(tX)_*\xi_Z \in \mathcal{A}_Z$. The former is immediate since X is left-invariant. The latter follows from Equation 5:

$$\begin{aligned}
\exp(tX)_*(\xi_Z|_x) =& \frac{d}{ds}\Big|_{s=0} \exp(tX)(\exp(s\xi_Z)(x)) \\
=& \frac{d}{ds}\Big|_{s=0} \exp(s\xi_Z)(\exp(tX)(x)) \\
=& \xi_Z|_{\exp(tX)(x)}.
\end{aligned}$$

□

Lemma 3.63. $\mathrm{Ham}(Z/G)$ *is a Lie subalgebra of* $\mathrm{Vect}(Z/G)$.

Proof. For any $f, g, h \in C^\infty(Z/G)$ and $a, b \in \mathbb{R}$, $\{af + bg, h\}_{z/G} = a\{f, h\}_{M/G} + b\{g, h\}_{Z/G}$, and so $aX_f + bX_g = X_{af+bg}$. Thus $\mathcal{H}(Z/G)$ is a real vector space. Next, the Jacobi identity for the Poisson bracket gives

$$\{\{f, g\}_{Z/G}, h\}_{Z/G} = -\{g, \{f, h\}_{Z/G}\}_{Z/G} + \{f, \{g, h\}_{Z/G}\}_{Z/G}.$$

This translates to

$$X_{\{f,g\}_{Z/G}} h = X_f X_g h - X_g X_f h = [X_f, X_g]h.$$

□

Orbital Maps. Similar to how we wish to replace strata with orbits of Lie algebras of vector fields, we wish to replace stratified maps with maps sending orbits into orbits.

In general, a smooth map between subcartesian spaces does not lift to a map between corresponding orbital tangent bundles. This is illustrated in the following example.

Example 3.64. Let $S = \{(x, y) \in \mathbb{R}^2 \mid xy = 0\}$, and let $\gamma : \mathbb{R} \to S$ be a curve passing through $(0, 0) \in S$ at time $t = 0$ such that

$$u := \frac{d}{dt}\Big|_{t=0} \gamma(t) \neq 0.$$

Then $u \notin \check{T}_{(0,0)}S$ since $\check{T}_{(0,0)}S = \{0\}$, but $\frac{d}{dt}\Big|_{t=0} \in \check{T}_0\mathbb{R}$.

To remedy this lack of the functoriality of $\check{T}$, we introduce a special kind of smooth map.

Definition 3.65. Let R and S be subcartesian spaces and let $F : R \to S$ be a smooth map between them. Let $\mathcal{F}$ and $\mathcal{G}$ be smooth families of vector fields on R and S, respectively. F is *orbital* with respect to $\mathcal{F}$ and $\mathcal{G}$ if for any $x \in R$, $F(O_x^{\mathcal{F}}) \subseteq O_{F(x)}^{\mathcal{G}}$. That is, for any $X \in \mathcal{F}$ and $x \in R$, there exist $Y_1, ..., Y_k \in \mathcal{G}$, $\epsilon > 0$ and $\epsilon_1, ..., \epsilon_k \geq 0$ such that for all $t \in (-\epsilon, \epsilon)$ there exist $t_i \in (-\epsilon_i, \epsilon_i)$ $(i = 1, ..., k)$ satisfying,

$$F(\exp(tX)(x)) = \exp(t_1 Y_1) \circ ... \circ \exp(t_k Y_k)(F(x)). \tag{6}$$

If $\mathcal{F} = \mathrm{Vect}(R)$ and $\mathcal{G} = \mathrm{Vect}(S)$, then we simply call F *orbital*.

Remark 3.66. Note that all charts and smooth functions on S are orbital, as are all diffeomorphisms between subcartesian spaces.

Proposition 3.67. *Let R and S be subcartesian spaces, and let F be an orbital map between them with respect to families of vector fields $\mathcal{F}$ on R and $\mathcal{G}$ on S. Then the restriction of the pushforward F_* to $\check{T}^{\mathcal{F}}R$ has image in $\check{T}^{\mathcal{G}}S$.*

Proof. Using the notation from the above definition, differentiating Equation 6 with respect to t and setting $t = 0$ gives us

$$F_*(X|_x) = Y|_{F(x)}.$$

□

The following theorem is a result of Schwarz; see [25] and [26] ([26] Chapter 1 Theorem 4.3 for full details). Let D be the Lie subgroup of $\mathrm{Diff}(M)^G$ consisting of G-equivariant diffeomorphisms of M that act trivially on $C^\infty(M)^G$ (that is, they send each G-orbit to itself), and let $\mathfrak{d}$ denote the Lie algebra of D.

Theorem 3.68 (Schwarz)**.** *The following is a split short exact sequence.*

$$0 \longrightarrow \mathfrak{d} \longrightarrow \mathrm{Vect}(M)^G \xrightarrow{\pi_*} \mathrm{Vect}(M/G) \longrightarrow 0 \tag{7}$$

Remark 3.69. Actually, Schwarz showed that π_* mapped $\mathrm{Vect}(M)^G$ onto stratified vector fields of M/G with its orbit-type stratification. But by Theorem 3.50, this family of vector fields is exactly $\mathrm{Vect}(M/G)$.

Remark 3.70. Since diffeomorphisms in D send G-orbits to themselves, we have that $\check{T}^{\mathfrak{d}}M$ is contained in $\check{T}^{\rho(\mathfrak{g})}M$.

Corollary 3.71. *The image of π_* restricted to $\check{T}^{\mathcal{A}}M$ is $\check{T}(M/G)$.*

Proof. π_* will map any vector in $\check{T}^{\rho(\mathfrak{g})}M$ to 0, and so it is enough to consider vectors in $\check{T}^{\mathcal{V}}M$ (where we set $\mathcal{V} := \mathrm{Vect}(M)^G$ for brevity). Let $x \in M$ and $v \in \check{T}^{\mathcal{V}}_x M$. Then, there exists a left-invariant vector field $X \in \mathcal{V}$ such that $X|_x = v$. By Theorem 3.68 there exists $Y \in \mathrm{Vect}(M/G)$ such that $Y|_{\pi(x)} = \pi_*(X|_x)$.

Now, let $w \in \check{T}_{\pi(x)}(M/G)$. There exists a vector field $Y \in \mathrm{Vect}(M/G)$ such that $Y|_{\pi(x)} = w$. Again by Theorem 3.68 there is a vector field $X \in \mathcal{V}$ such that $\pi_* X = Y$, and so $\pi_*(X|_x) = w$. □

Corollary 3.72. *π is orbital with respect to $\mathcal{A}$ and* $\mathrm{Vect}(M/G)$*.*

Proof. Since π_* will map any vector field in $\rho(\mathfrak{g})$ to the zero vector field on M/G, and local flows of $\mathrm{Vect}(M)^G$ and $\rho(\mathfrak{g})$ commute, it is enough to check that π is orbital with respect to $\mathrm{Vect}(M)^G$ and $\mathrm{Vect}(M/G)$. Let $X \in \mathrm{Vect}(M)^G$. Then by Theorem 3.68, there is a vector field $Y \in \mathrm{Vect}(M/G)$ such that $\pi_* X = Y$. Fix $x \in M$. Then

$$\frac{d}{dt}\pi(\exp(tX)(x)) = \pi_*(X|_x) = Y|_{\pi(x)} = \frac{d}{dt}\exp(tY)(\pi(x)).$$

By the ODE theorem, we have that

$$\pi(\exp(tX)(x)) = \exp(tY)(\pi(x))$$

for all t where it is defined. Hence orbits in $\mathcal{O}_{\mathcal{A}}$ are mapped via π to orbits of M/G induced by $\mathrm{Vect}(M/G)$. ☐

Corollary 3.73. *A local flow of M/G lifts to a G-equivariant local flow of M.*

Proof. Fix a vector field $Y \in \mathrm{Vect}(M/G)$. By Theorem 3.68 there is a vector field $X \in \mathrm{Vect}(M)^G$ such that $\pi_* X = Y$. From the ODE theorem we have that

$$\pi(\exp(tX)(x)) = \exp(tY)(\pi(x))$$

for all $x \in M$ and $t \in I^X_x$. ☐

Lemma 3.74. *$\mathcal{O}_{\mathcal{A}}$ induces the orbit-type stratification of M with respect to G.*

Proof. Fix $x \in M$, and let $H \leq G$ be a closed subgroup of G such that $x \in M_{(H)}$. Now let $y \in O^{\mathcal{A}}_x$. Then, there exist vector fields $X_1, ..., X_k \in \mathcal{A}$ and $t_1, ..., t_k \in \mathbb{R}$ such that

$$y = \exp(t_1 X_1) \circ ... \circ \exp(t_k X_k)(x).$$

But then, by Corollary 3.72 and Theorem 3.68, there exist $Y_1, ..., Y_k \in \mathrm{Vect}(M/G)$ such that

$$\pi(y) = \exp(t_1 Y_1) \circ ... \circ \exp(t_k Y_k)(\pi(x)).$$

Hence, $\pi(x)$ and $\pi(y)$ are in the same orbit $O_{\pi(x)}$. But this is a stratum of the orbit-type stratification of M/G by Theorem 3.50, and so $y \in M_{(H)}$. Thus $O^{\mathcal{A}}_x \subseteq M_{(H)}$.

Now, let z be a point in the same connected component of $M_{(H)}$ as x. Then again by Theorem 3.50, $\pi(y)$ and $\pi(x)$ are in the same orbit $O_{\pi(x)}$, and hence there exist vector fields $Y_1, ..., Y_k$ and $t_1, ..., t_k \in \mathbb{R}$ such that $\pi(y) = \exp(t_1 X_1) \circ ... \circ \exp(t_k X_k)(\pi(x))$. By Corollary 3.73, there are vector fields $X_1, ..., X_k \in \mathcal{A}$ such that $y = \exp(t_1 X_1) \circ ... \circ \exp(t_k X_k)(x)$. ☐

Lemma 3.75. *i is orbital with respect to $\mathcal{A}_Z$ and $\mathcal{A}$.*

Proof. Let $X \in \mathcal{A}_Z$ and fix $z \in Z \subseteq M$. Then by Lemma 3.56 there exist a G-invariant open neighbourhood $U \subseteq M$ of z and $\tilde{X} \in \mathcal{A}$ such that $X|_{U \cap Z} = U \tilde{\cap} Z$. Applying the ODE theorem, we are done. ☐

Corollary 3.76. *$\bigwedge^k_Z \check{T}^{\mathcal{A}_Z} Z$ is a differential subspace of $\bigwedge^k_M \check{T}^{\mathcal{A}} M$ for each k.*

Proof. This is immediate from Lemma 3.75 and Proposition 3.67. ☐

Lemma 3.77. *π_Z is orbital with respect to $\mathcal{A}_Z$ and* $\mathrm{Vect}(Z/G)$.

Proof. By Lemma 3.62, it is enough to show this separately for $\rho_Z(\mathfrak{g})$ and $\mathrm{Vect}(Z)^G$. For the first subalgebra,

$$\pi(\exp(t\xi_Z)(z)) = \pi(z) = \exp(0)(\pi(z))$$

for all $z \in Z$ and t for which the integral curve is defined.

Now fix $X \in \mathrm{Vect}(Z)^G$. Using Lemma 3.56 cover Z/G with a locally finite open cover $\{V_\alpha\}_{\alpha \in A}$ such that for every $\alpha \in A$, there exist $\tilde{X}^\alpha \in \mathrm{Vect}(M)^G$ satisfying $i_*(X|_{\pi_Z^{-1}(V_\alpha)}) = \tilde{X}^\alpha|_{\pi_Z^{-1}(j(V_\alpha))}$. Note that for any $\alpha \in A$, $x \in V_\alpha$, $z \in \pi_Z^{-1}(x)$ and $f \in \mathfrak{n}(j(Z/G))$,

$$\begin{aligned}(\pi_* \tilde{X}^\alpha)|_{j(x)} f =& \tilde{X}^\alpha|_{i(z)} \pi^* f \\ =& X|_z i^* \pi^* f \\ =& X|_z \pi_z^* j^* f \\ =& 0.\end{aligned}$$

Let $\{\zeta_\alpha\}_{\alpha \in A}$ be a partition of unity subordinate to $\{V_\alpha\}$, and for each $\alpha \in A$, let $\tilde{\zeta}_\alpha$ be an extension of ζ_α to M/G. Define

$$\tilde{Y} := \sum_\alpha \tilde{\zeta}_\alpha (\pi_* \tilde{X}^\alpha)|_{j(Z/G)}.$$

From the above, we have that $\tilde{Y} \in \mathrm{Der}C^\infty(j(Z/G))$, and so in particular, $\tilde{Y}$ pullsback to a global derivation $Y \in \mathrm{Der}C^\infty(S)$. Also, for any $z \in Z$,

$$\begin{aligned}j_* \pi_{Z*}(X|_z) =& \sum_\alpha \tilde{\zeta}_\alpha j_* \pi_{Z*}(X|_z) \\ =& \sum_\alpha \tilde{\zeta}_\alpha \pi_*(\tilde{X}^\alpha|_z) \\ =& \tilde{Y}|_{\pi_Z(z)}.\end{aligned}$$

Thus, $\pi_{Z*}(X|_z) = Y|_{\pi_Z(z)}$. Finally, we need to show that Y is a vector field, and we shall do so by appealing to Proposition 3.17. Fix $z \in Z$, and define $\gamma(t) := \pi_Z(\exp(tX)(z))$. Differentiating, we see that γ is an integral curve of Y through $\pi_Z(z)$. But γ has an open domain and π_Z is surjective, and so γ is maximal. Thus Y is a vector field. ☐

Lemma 3.78. *j is orbital with respect to* $\mathrm{Ham}(Z/G)$ *and* $\mathrm{Vect}(M/G)$.

Proof. By Lemma 3.51 orbits of $\mathrm{Ham}(Z/G)$ are contained in the orbit-type strata of Z/G, which in turn are contained in the orbit-type strata of M/G. By Theorem 3.50, connected components of the orbit-type strata of M/G are the orbits induced by $\mathrm{Vect}(M/G)$. ☐

Summary Thus Far. Our setting can now be described as follows. We have the following commutative diagramme in the category of subcartesian spaces where i is orbital with respect to $\mathcal{A}_Z$ and $\mathcal{A}$, π is orbital with respect to $\mathcal{A}$ and $\mathrm{Vect}(M/G)$, π_Z is orbital with respect to $\mathcal{A}_Z$ and $\mathrm{Vect}(Z/G)$, and finally j is orbital with respect to $\mathrm{Ham}(Z/G)$ and $\mathrm{Vect}(M/G)$.

$$
\begin{tikzcd}
Z \arrow[r, "i"] \arrow[d, "\pi_Z"'] & M \arrow[d, "\pi"] \\
Z/G \arrow[r, "j"'] & M/G
\end{tikzcd}
$$

4. Differential Forms

Orbital Differential Forms. Fix a locally complete Lie subalgebra $\mathfrak{a}$ of $\mathrm{Vect}(S)$.

Definition 4.1. An $\mathfrak{a}$-*orbital (differential) k-form* μ on S is a smooth fibrewise-linear function from $\bigwedge_S^k \check{T}^{\mathfrak{a}}S = \bigcup_{x\in S}\bigwedge^k \check{T}_x^{\mathfrak{a}}S$ to $\mathbb{R}$. Denote the set of all such k-forms by $\check{\Omega}_{\mathfrak{a}}^k(S)$. The wedge product works as it does in the theory of smooth manifolds, and so we have the corresponding graded exterior algebra $\check{\Omega}_{\mathfrak{a}}(S) := \bigoplus_{i=0}^{\infty}\check{\Omega}_{\mathfrak{a}}^i(S)$ with $\check{\Omega}_{\mathfrak{a}}^0(S) = C^\infty(S)$. $C^\infty(S)$ acts linearly on $\check{\Omega}_{\mathfrak{a}}(S)$ in the obvious way. For $\mu \in \check{\Omega}_{\mathfrak{a}}^k(S)$ and $v_1, ..., v_k \in \check{T}_x^{\mathfrak{a}}S$, we write $\mu(v_1, ..., v_k)$ for $\mu(v_1 \wedge ... \wedge v_k)$.

Note that we can evaluate an $\mathfrak{a}$-orbital differential form on vector fields in $\mathfrak{a}$ as well: if $X_1, ..., X_k \in \mathfrak{a}$ and $\mu \in \check{\Omega}_{\mathfrak{a}}^k(S)$, then for any $x \in S$,

$$\mu(X_1, ..., X_k)(x) := \mu(X_1|_x, ..., X_k|_x).$$

This is a smooth function on S, as it is a composition of smooth functions.

Definition 4.2. Let R and S be subcartesian spaces and let $\mathfrak{a}$ and $\mathfrak{b}$ be locally complete Lie subalgebras of $\mathrm{Vect}(R)$ and $\mathrm{Vect}(S)$, respectively. Let $F : R \to S$ be an orbital map with respect to $\mathfrak{a}$ and $\mathfrak{b}$. If $\mu \in \check{\Omega}_{\mathfrak{b}}^k(S)$, then $F^*\mu \in \check{\Omega}_{\mathfrak{a}}^k(R)$ is defined by

$$F^*\mu(v_1, ..., v_k) = \mu(F_*v_1, ..., F_*v_k).$$

Orbital k-forms are smooth functions on $\bigwedge_S^k \check{T}^{\mathfrak{a}}S$, and so they have local representatives on Euclidean space. We will show that these representatives can be chosen to be differential forms. For an orbital k-form μ and a subset $U \subseteq S$, denote by $\mu|_U$ the restriction $\mu|_{\bigwedge_S^k \check{T}^{\mathfrak{a}}S|_U}$. Let $f \in C^\infty(S)$, and let $v \in \check{T}^{\mathfrak{a}}S$. Define $df(v) := v(f)$. The $df|_{\check{T}^{\mathfrak{a}}S}$ is an $\mathfrak{a}$-orbital 1-form.

Lemma 4.3. *Let $\mathfrak{a}$ be a locally complete Lie subalgebra of* $\mathrm{Vect}(S)$. *Let $\mu \in \check{\Omega}_{\mathfrak{a}}^k(S)$. Then for each $x \in S$, there exist a chart $\varphi : V \to \tilde{V} \subseteq \mathbb{R}^n$ about x and $\tilde{\mu} \in \Omega^k(\mathbb{R}^n)$ such that $\mu|_V = \varphi^*\tilde{\mu}$.*

Proof. Recall that for any chart $\varphi : V \to \tilde{V} \subseteq \mathbb{R}^n$ on S, φ_* will be a chart on $\bigwedge_S^k \check{T}^{\mathfrak{a}}S$ that sends fibres over V linearly into fibres of $\bigwedge_{\mathbb{R}^n}^k T\mathbb{R}^n$. Thus, for any $v \in \bigwedge_S^k \check{T}^{\mathfrak{a}}S$, there is a chart $\varphi : V \to \tilde{V} \subseteq \mathbb{R}^n$ about $\check{\tau}_{\mathfrak{a}}(v)$ and a smooth function $f : \bigwedge_{\mathbb{R}^n}^k T\mathbb{R}^n \to \mathbb{R}$ such that $\mu|_V = \varphi^* f$ (as functions). Thus we only need to show that we can choose f to be a k-form on $\mathbb{R}^n$.

For brevity, let $E := \bigwedge_S^k \check{T}^{\mathfrak{a}}S$. Fix a fibre E_x in E. Since E_x is a linear space, the vertical tangent space of E_x at 0, denoted $T(E_x)|_0$, is linearly isomorphic to E_x in a canonical way. Define $i : E \to TE : v \mapsto \hat{v}$ sending $v \in E_{\check{\tau}_{\mathfrak{a}}(v)}$ to the corresponding vector in $T(E_{\check{\tau}_{\mathfrak{a}}(v)})|_0$. Fix a chart $\varphi : V \to \tilde{V} \subseteq \mathbb{R}^n$ and a smooth function $f : \tilde{E} \to \mathbb{R}$ such that $\mu|_V = \varphi^* f$. Let E_V denote $\check{\tau}_{\mathfrak{a}}^{-1}(V)$, $\tilde{E} := \bigwedge_{\mathbb{R}^n}^k T\mathbb{R}^n$ and define $\tilde{i} : \tilde{E} \to T\tilde{E}$ similarly to i. Also, denote by $TE_V|_0$ and $T\tilde{E}|_0$ the restrictions of the (Zariski) tangent bundles of E_V and $\tilde{E}$ to their zero sections, respectively. Note that for any $x \in V$, $\varphi_* : E_x \to \tilde{E}_{\varphi(x)}$ lifts to a linear map $\tilde{\varphi}_* : TE_x|_0 \to T\tilde{E}_{\varphi(x)}|_0$ such that the following diagramme commutes.

$$\begin{tikzcd}
TE_V|_0 \arrow[r, "\tilde{\varphi}_*"] & T\tilde{E}|_0 \arrow[rr, "(\partial_{vert} f)|_0"] & & \mathbb{R} \\
E_V \arrow[u, "i"] \arrow[r, "\varphi_*"'] \arrow[d, "\check{\tau}_{\mathfrak{a}}"'] & \tilde{E} \arrow[u, "\tilde{i}"'] \arrow[rr, "f"] \arrow[d, "\tau"'] & & \mathbb{R} \\
V \arrow[r, "\varphi"'] & \mathbb{R}^n
\end{tikzcd}$$

Here ∂_{vert} denotes the differential in the vertical direction; that is, along the fibre. So, since μ is a fibrewise linear map, we have $\mu(v) = (\partial_{vert}\mu)|_0(\hat{v})$. But since $\mu|_V = f \circ \varphi_*$, by the chain rule we have

$$(\partial_{vert}\mu)|_0 = (\partial_{vert})f|_0 \circ \tilde{\varphi}_*.$$

Combining these two facts, we have for any $v \in E_V$,

$$\begin{aligned}
\mu|_V(v) =& (\partial_{vert}\mu)|_0(\hat{v}) \\
=& (\partial_{vert} f)|_0(\tilde{\varphi}_*(\hat{v})) \\
=& (\partial_{vert} f)|_0(\tilde{i}(\varphi_*(v))).
\end{aligned}$$

Since $(\partial_{vert} f)|_0 \circ \tilde{i}$ is a fibrewise linear map on $\tilde{E}$ whose pullback via φ_* is equal to $\mu|_V$, we set $\tilde{\mu} = (\partial_{vert} f)|_0 \circ \tilde{i}$ and we are done. □

Remark 4.4. A more general result involving tensor fields on "vector pseudo-bundles" was proved by Marshall in [17], and the above proof is based on this.

Remark 4.5. If $\mathfrak{a}$ and $\mathfrak{b}$ are locally complete Lie subalgebras of $\mathrm{Vect}(S)$ such that $\mathfrak{a} \subseteq \mathfrak{b}$, then we obtain a restriction map $r : \check{\Omega}_{\mathfrak{b}}(S) \to \check{\Omega}_{\mathfrak{a}}(S)$. The construction of $\tilde{\mu}$ in the proof above is independent of $\mathfrak{a}$, and hence also represents a "local extension" of μ in $\check{\Omega}^k_{\mathfrak{b}}(S)$. To show that this works globally, we need the following lemma.

Lemma 4.6. *Let $\varphi : V \to \tilde{V} \subseteq \mathbb{R}^m$ and $\psi : V \to \tilde{W} \subseteq \mathbb{R}^n$ be two charts on S. For $\mu \in \check{\Omega}^k_{\mathfrak{a}}(S)$, the map $\psi \circ \varphi^{-1}$ takes a local representative $\tilde{\mu}_\psi \in \Omega^k(\mathbb{R}^n)$ of μ with respect to ψ to a local representative $\tilde{\mu}_\varphi \in \Omega^k(\mathbb{R}^m)$ of μ with respect to φ.*

Proof. By Lemma 2.29 there is a smooth map $\zeta : \mathbb{R}^m \to \mathbb{R}^n$ such that $\zeta|_{\tilde{V}} = \psi \circ \varphi^{-1}$. So,

$$\varphi^*(\zeta^*\tilde{\mu}_\psi) = \psi^*\tilde{\mu}_\psi = \mu|_V.$$

Define $\tilde{\mu}_\varphi := \zeta^*\tilde{\mu}_\psi \in \Omega^k(\mathbb{R}^m)$. □

Thus, in the notation of Remark 4.5, for any $\mu \in \mathfrak{a}$, about any point we can find a local representative $\tilde{\mu}$ which is a form on $\mathbb{R}^n$. Using a partition of unity, we can put the pullbacks of these local representatives together to form a global form $\mu' \in \mathfrak{b}$ whose restriction to vectors in $\check{T}^{\mathfrak{a}}S$ is μ. Note, however, that in general this extension is not unique, unless we impose a condition on $\mathfrak{a}$.

Proposition 4.7. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be locally complete Lie subalgebras of* $\mathrm{Vect}(S)$ *such that $\mathfrak{a} \subseteq \mathfrak{b}$ and such that there is an open dense orbit in $\mathcal{O}_{\mathfrak{a}}$. Then, $\mu \in \check{\Omega}^k_{\mathfrak{a}}(S)$ extends uniquely to $\mu' \in \check{\Omega}^k_{\mathfrak{b}}(S)$, and $r : \check{\Omega}_{\mathfrak{b}}(S) \to \check{\Omega}_{\mathfrak{a}}(S)$ is an isomorphism of exterior algebras.*

Proof. By the above discussion, there exists an extension $\mu' \in \check{\Omega}^k_{\mathfrak{b}}(S)$ of μ. Since $\mathfrak{a} \subseteq \mathfrak{b}$, by definition of an orbit, we have that an open dense orbit $U \in \mathcal{O}_{\mathfrak{a}}$ is contained in an open dense orbit contained in $\mathcal{O}_{\mathfrak{b}}$. For any $x \in U$, $\check{T}^{\mathfrak{a}}_x S = \check{T}^{\mathfrak{b}}_x S$, and so $\mu' = \mu$ on U. Note that $\check{T}^{\mathfrak{b}} S$ is locally trivial over U, and applying Corollary 3.41, we have that $\check{\tau}_{\mathfrak{b}}^{-1}(U) = \check{\tau}_{\mathfrak{a}}^{-1}(U)$ is open and dense in $\check{T}^{\mathfrak{b}} S$. By continuity, μ' is uniquely determined by its restriction to U. Since this restriction is equal to μ on U, we have that μ uniquely determines μ'. This extension commutes with the restriction of μ' to $\check{T}^{\mathfrak{a}} S$, and so we have an isomorphism between $\check{\Omega}^k_{\mathfrak{a}}(S)$ and $\check{\Omega}^k_{\mathfrak{b}}(S)$. $\square$

Henceforth when we refer to a local representative of a k-form on S with respect to some chart $\varphi : U \to \tilde{U} \subseteq \mathbb{R}^n$, we shall mean a k-form on $\mathbb{R}^n$.

The Exterior Derivative.

Definition 4.8. Let S be a subcartesian space and $\mathfrak{a} \subseteq \mathrm{Vect}(S)$ a locally complete Lie subalgebra. The *exterior derivative* on $\check{\Omega}_{\mathfrak{a}}(S)$ is the map $d : \check{\Omega}^k_{\mathfrak{a}}(S) \to \check{\Omega}^{k+1}_{\mathfrak{a}}(S)$ such that for any $x \in S$, and any chart $\varphi : U \to \tilde{U} \subseteq \mathbb{R}^n$ about x, and any $\mu \in \check{\Omega}^k_{\mathfrak{a}}(S)$ with local representative $\tilde{\mu}$ with respect to φ,

$$d\mu = \varphi^* d\tilde{\mu}.$$

Lemma 4.9. *The exterior derivative is well-defined; that is, it is independent of the chart and local representative chosen.*

Proof. Using the same notation as above, let $\tilde{\mu}'$ be another local representative of μ with respect to φ. Let $v_1, ..., v_{k+1} \in \check{T}^{\mathfrak{a}}_x S$, and let $X_1, ..., X_{k+1} \in \mathfrak{a}$ such that $X_i|_x = v_i$ for each $i = 1, ..., k+1$. Let $\tilde{X}_i$ be a local extension of X_i with respect to φ. Then,

$$\begin{aligned}
&\varphi^* d(\tilde{\mu} - \tilde{\mu}')(v_1, ..., v_{k+1})\\
=&d(\tilde{\mu} - \tilde{\mu}')(\tilde{X}_1, ..., \tilde{X}_{k+1})(\varphi(x))\\
=&\sum_{i=1}^{k+1}(-1)^{i+1}\tilde{X}_i((\tilde{\mu} - \tilde{\mu}')(\tilde{X}_1, ..., \widehat{\tilde{X}_i}, ..., \tilde{X}_{k+1}))(\varphi(x))\\
&+ \sum_{1\le i<j\le k+1}(-1)^{i+j}(\tilde{\mu} - \tilde{\mu}')([\tilde{X}_i, \tilde{X}_j], ..., \widehat{\tilde{X}_i}, ..., \widehat{\tilde{X}_j}, ..., \tilde{X}_{k+1})(\varphi(x))\\
=&\sum_{i=1}^{k+1}(-1)^{i+1}X_i((\varphi^*(\tilde{\mu} - \tilde{\mu}'))(X_1|_x, ..., \widehat{X_i|_x}, ..., X_{k+1}|_x))\\
&+ \sum_{1\le i<j\le k+1}(-1)^{i+j}\varphi^*(\tilde{\mu} - \tilde{\mu}')([X_i, X_j]|_x, ..., \widehat{X_i|_x}, ..., \widehat{X_j|_x}, ..., X_{k+1}|_x).
\end{aligned}$$

Since $\varphi^*(\tilde{\mu} - \tilde{\mu}')$ vanishes on $\check{T}^{\mathfrak{a}} S$, the terms in the first sum vanish by Proposition 3.6; hence each term is 0. Since $\mathfrak{a}$ is closed under the Lie bracket and $\varphi^*(\tilde{\mu} - \tilde{\mu}')$ vanishes on $\check{T}^{\mathfrak{a}} S$, the second sum also is 0.

If $\psi : U \to \tilde{V} \subseteq \mathbb{R}^m$ is another chart, then by Lemma 2.29 there exists a smooth map $\zeta : \mathbb{R}^m \to \mathbb{R}^n$ such that $\zeta|_{\tilde{U}} = \varphi \circ \psi^{-1}$. Thus, $\zeta^* \tilde{\mu}$ is a local representative of μ with respect to ψ, and $\psi^* d(\zeta^* \tilde{\mu}) = \varphi^* d\tilde{\mu}$. $\square$

Corollary 4.10. *The (unique) extension map in Proposition 4.7 commutes with d.*

Proof. This is a direct consequence of the lemma above and the definition of the exterior derivative on orbital differential forms. □

Proposition 4.11.
$$d^2 = 0$$

Proof. It is sufficient to prove this locally. Let $\varphi : U \to \tilde{U} \subseteq \mathbb{R}^n$ be a chart. Let $\mu \in \check{\Omega}^k_{\mathfrak{a}}(S)$ and let $\tilde{\mu} \in \Omega^k(\mathbb{R}^n)$ be a local representative of μ with respect to φ. Then,
$$(d^2\mu)|_U = d(\varphi^* d\tilde{\mu}) = \varphi^* d^2\tilde{\mu} = 0.$$
□

Lemma 4.12. *Let R and S be subcartesian spaces with locally complete Lie subalgebras $\mathfrak{a}$ and $\mathfrak{b}$ of $\mathrm{Vect}(R)$ and $\mathrm{Vect}(S)$ respectively. Let $F : R \to S$ be an orbital map with respect to $\mathfrak{a}$ and $\mathfrak{b}$. Fix $x \in R$, let $\psi : V \to \tilde{V} \subseteq \mathbb{R}^n$ be a chart about $F(x)$ and $\varphi : U \to \tilde{U} \subseteq \mathbb{R}^m$ be a chart about x such that $U \subseteq F^{-1}(V)$. Let $\tilde{F} \in C^\infty(\mathbb{R}^m, \mathbb{R}^n)$ such that $F|_U = \psi^{-1} \circ \tilde{F} \circ \varphi$. Fix $\mu \in \check{\Omega}^k_{\mathfrak{b}}(S)$ and let $\tilde{\mu}$ be a local representative of μ with respect to ψ. Then $\tilde{F}^*\tilde{\mu}$ is a local representative of $F^*\mu$ with respect to φ.*

Proof.
$$(F^*\mu)|_U = (F^*\psi^*\tilde{\mu})|_U = (\varphi^*\tilde{F}^*\tilde{\mu})|_U.$$
□

Corollary 4.13. *Using the same setting as Lemma 4.12,*
$$d(F^*\mu) = F^* d\mu.$$

Proof. It is enough to check the equality locally.
$$\begin{aligned}(dF^*\mu)|_U =& \varphi^*(d\tilde{F}^*\tilde{\mu})\\ =& \varphi^*(\tilde{F}^* d\tilde{\mu})\\ =& F^*\psi^* d\tilde{\mu}\\ =& F^*((d\mu)|_V).\end{aligned}$$
□

Differential Forms on M and M/G. We return once more to Diagramme 1. The goal of this subsection is to prove the following theorem.

Theorem 4.14. $\pi^* : (\check{\Omega}(M/G), d) \to (\Omega_{basic}(M), d)$ *is an isomorphism of complexes.*

Remark 4.15. Since the complex of basic forms is isomorphic to $(\check{\Omega}(M/G), d)$, it is a smooth invariant of M/G.

Note by Corollary 3.72 that π^* sends forms in $\check{\Omega}^k(M/G)$ to forms in $\check{\Omega}^k_{\mathcal{A}}(M)$, and so we need to make sense of the map in the statement of Theorem 4.14. The following proposition does exactly this.

Proposition 4.16. $(\check{\Omega}_{\mathcal{A}}(M), d)$ *is isomorphic to* $(\Omega(M), d)$ *as a complex.*

Proof. By Lemma 3.74, $\mathcal{O}_{\mathcal{A}}$ has an open dense orbit $M_{(K)}$ for some closed subgroup $K \leq G$. By Proposition 4.7 and Corollary 4.10, we have that $(\check{\Omega}_{\mathcal{A}}(M), d) \cong (\check{\Omega}(M), d)$. Since M is a manifold, we have that $(\check{\Omega}(M), d) = (\Omega(M), d)$, and this proves the proposition. $\square$

Next we show that the image of π^* on $\check{\Omega}^k(M/G)$ is contained in basic forms.

Lemma 4.17. *For each $\mu \in \check{\Omega}^k(M/G)$, $\pi^*\mu$ extends uniquely to a basic form on M.*

Proof. Fix $\mu \in \check{\Omega}^k(M/G)$. Then since $(\check{\Omega}_{\mathcal{A}}(M), d) \cong (\Omega(M), d)$, we know that there is a unique extension $\tilde{\mu} \in \Omega^k(M)$ of $\pi^*\mu$ such that $\tilde{\mu}|_{\bigwedge_M^k \check{T}^{\mathcal{A}}M} = \pi^*\mu$. We want to show that $\tilde{\mu}$ is basic.

Let us begin by showing that $\tilde{\mu}$ is horizontal on $\check{\tau}_{\rho(\mathfrak{g})}^{-1}(M_{(K)})$, which is open and dense in $\check{T}^{\rho(\mathfrak{g})}M$ by Corollary 3.41. Fix $x \in M_{(K)}$, and let $v_1, ..., v_k \in T_xM = \check{T}_x^{\mathcal{A}}M$. If $v_l \in \check{T}_x^{\rho(\mathfrak{g})}M$ for some $l = 1, ..., k$, then

$$\tilde{\mu}(v_1, ..., v_k) = \mu(\pi_*v_1, ..., \pi_*v_k) = 0$$

since $\pi_*v_l = 0$. By continuity, $\tilde{\mu}$ is horizontal on all of M. To check G-invariance, again let $x \in M_{(K)}$ and $v_1, ..., v_k \in T_xM = \check{T}_x^{\mathcal{A}}M$. Then, for any $g \in G$,

$$\begin{aligned} g^*\tilde{\mu}(v_1, ..., v_k) =& \pi^*\mu(g_*v_1, ..., g_*v_k) \\ =& \mu(\pi_*g_*v_1, ..., \pi_*g_*v_k) \\ =& \mu((\pi \circ g)_*v_1, ..., (\pi \circ g)_*v_k) \\ =& \mu(\pi_*v_1, ..., \pi_*v_k) \\ =& \tilde{\mu}(v_1, ..., v_k). \end{aligned}$$

Now we will extend this result to all of M. Let $x \in M$ and $v_1, ..., v_k \in T_xM$. Then since $\check{\tau}_{\mathcal{A}}^{-1}(M_{(H)})$ is open and dense in $\check{T}^{\mathcal{A}}M$, there exists a sequence $\{x_j\} \subset M_{(K)}$ that converges to x and sequences of vectors $\{v_i^j\} \subset T_{x_j}M = \check{T}_{x_j}^{\mathcal{A}}M$ that converge to v_i for each i. By continuity, we have

$$\begin{aligned} g^*\tilde{\mu}(v_1, ..., v_k) =& \lim_{j\to\infty} g^*\tilde{\mu}(v_1^j, ..., v_k^j) \\ =& \lim_{j\to\infty} \tilde{\mu}(v_1^j, ..., v_k^j) \\ =& \tilde{\mu}(v_1, ..., v_k). \end{aligned}$$

Thus, $\tilde{\mu}$ is basic. This establishes that $\pi^* : \check{\Omega}^k(M/G) \to \Omega_{basic}^k(M)$ is a well-defined map of $\mathbb{R}$-algebras for each k. $\square$

Lemma 4.18. *$\pi^* : \check{\Omega}^k(M/G) \to \Omega_{basic}^k(M)$ is injective.*

Proof. Let $\mu \in \check{\Omega}^k(M/G)$ such that $\pi^*(\mu) = 0$. Then for any $v_1, ..., v_k \in \check{T}_x^{\mathcal{A}}M$,

$$\mu(\pi_*v_1, ..., \pi_*v_k) = 0.$$

By Corollary 3.71, π_* is surjective onto $\check{T}(M/G)$, and so $\mu = 0$. Hence π^* is injective. $\square$

Definition 4.19. Let $\Omega^k(M)^G$ be the space of G-invariant k-forms on M. We say that $\tilde{\mu} \in \Omega^k(M)^G$ is *basic at x* if for any $v \in \check{T}_x^{\rho(\mathfrak{g})}M$, $v \lrcorner \tilde{\mu} = 0$.

Lemma 4.20. *For any $k \geq 0$, the following short exact sequence is split:*

$$0 \longrightarrow \check{\Omega}^k(M/G) \xrightarrow{\pi^*} \Omega^k(M)^G \longrightarrow \Omega^k(M)^G/\pi^*(\check{\Omega}^k(M/G)) \longrightarrow 0.$$

In particular, there is an $\mathbb{R}$-algebra homomorphism $\iota : \Omega^k(M)^G \to \check{\Omega}^k(M/G)$ such that $\iota \circ \pi^*$ is the identity map on $\check{\Omega}^k(M/G)$. Moreover, if $\tilde{\mu} \in \Omega^k(M)^G$ is basic at $x \in M$, then $\iota(\tilde{\mu})|_{\check{T}_x^{\mathcal{A}}M} = \tilde{\mu}|_{\check{T}_x^{\mathcal{A}}M}$.

Proof. It is enough to show that this holds locally. We return to the setting of Diagramme 2 in the proof of Theorem 2.30. Note that if σ is a G-invariant polynomial on $\mathbb{R}^N$, then both $\sigma \circ \tau$ and $d\sigma$ are G-invariant polynomials on $T\mathbb{R}^N$, where $\tau : T\mathbb{R}^N \to \mathbb{R}^N$ is the usual projection. Recalling that $\{p_1, ..., p_k\}$ is a generating set for $\mathbb{R}[x^1, ..., x^N]^G$, we can apply the result of Schwarz [24] again: there exists $l \geq 2k$ and G-invariant polynomials $\sigma_1, ..., \sigma_{l-2k}$ on $T\mathbb{R}^N \cong \mathbb{R}^{2N}$ such that $\{p_1 \circ \tau, ..., p_k \circ \tau, dp_1, ..., dp_k, \sigma_1, ..., \sigma_{l-2k}\}$ generates the G-invariant polynomials on $T\mathbb{R}^N$, and the map $p_T : T\mathbb{R}^N \to \mathbb{R}^l$ defined by

$$p_T = (p_1 \circ \tau, ..., p_k \circ \tau, dp_1, ..., dp_k, \sigma_1, ..., \sigma_{l-2k})$$

descends to a smooth embedding $(TU)/G \hookrightarrow \mathbb{R}^l$. Set $z : \mathbb{R}^N \to T\mathbb{R}^N$ to be the zero section. Note that for any $j = 1, ..., l-2k$, we have that $z^*\sigma_j$ is a G-invariant polynomial on $\mathbb{R}^N$, and so there exists a polynomial $\tilde{\sigma}_j : \mathbb{R}^k \to \mathbb{R}$ such that $z^*\sigma_j = \tilde{\sigma}_j(p_1, ..., p_k)$ for each $j = 1, ..., l-2k$. Define $\tilde{z} : \mathbb{R}^k \to \mathbb{R}^l$ by

$$\tilde{z}(x^1, ..., x^k) = (x^1, ..., x^k, 0, ..., 0, \tilde{\sigma}_1, ..., \tilde{\sigma}_{l-2k}).$$

Then $\tilde{z} \circ p = p_T \circ z$.

Next, $dz : T\mathbb{R}^N \to \mathrm{vert}|_0$ is a linear isomorphism, where $\mathrm{vert}|_0$ is an abbreviation for the vertical tangent bundle of $T\mathbb{R}^N$ restricted to the image of the zero section. Also, the image of $dp_T|_0 := dp_T|_{z(\mathbb{R}^N)}$ is contained in $T\mathbb{R}^l|_{\tilde{z}(\mathbb{R}^k)}$. Now that we are done with the setting, we will prove the theorem for $k = 1$. The higher exterior powers are proved similarly, replacing tangent bundles with fibred exterior powers of the tangent bundles. Fix $\tilde{\mu} \in \Omega^1(M)^G$. Then, this is a smooth G-invariant function on $T\mathbb{R}^N$, and by Schwarz there exists a smooth function η on $\mathbb{R}^l$ such that $\eta \circ p_T = \tilde{\mu}$. We can summarise everything above in the following commutative diagramme.

$$\begin{tikzcd}
T\mathbb{R}^N \arrow[r, "dz"] \arrow[d] \arrow[dr, equal] \arrow[ddd, bend right, "dp"'] & \mathrm{vert}|_0 \arrow[r, "\partial_{vert}\tilde{\mu}|_0"] \arrow[d] & \mathbb{R} \\
\mathbb{R}^N \arrow[r, "z"'] \arrow[d, "p"] & T\mathbb{R}^N \arrow[r, "\tilde{\mu}"] \arrow[d, "p_T"'] & \mathbb{R} \\
\mathbb{R}^k \arrow[r, "\tilde{z}"'] & \mathbb{R}^l \arrow[ur, bend right, "\eta"'] & \\
T\mathbb{R}^k \arrow[r, "d\tilde{z}"'] \arrow[u] & T\mathbb{R}^l|_{\tilde{z}(\mathbb{R}^k)} \arrow[r, "d\eta|_{\tilde{z}(\mathbb{R}^k)}"'] \arrow[u] & \mathbb{R}
\end{tikzcd}$$

Now, since $\tilde{\mu}$ is linear on the fibres of $T\mathbb{R}^N$, it is equal to $\partial_{vert}\tilde{\mu}|_0 \circ dz$ where ∂_{vert} is the derivative in the vertical (fibre) direction. But $\partial_{vert}(\tilde{\mu})|_0 = d\eta|_{\tilde{z}(\mathbb{R}^k)} \circ \partial_{vert}p_T|_0$. Note that

$\partial_{vert} p_T|_0 \circ dz = dp_T|_{z(\mathbb{R}^N)} \circ dz$. Since $p_T \circ z = \tilde{z} \circ p$, we thus have

$$\tilde{\mu} = d\eta|_{\tilde{z}(\mathbb{R}^k)} \circ d\tilde{z} \circ dp. \tag{8}$$

Now, if $\tilde{p}$ is the inclusion of $\mathbb{R}^N/G$ into $\mathbb{R}^k$ as in Diagramme 2, then $d\tilde{p}$ restricts to an inclusion of $\check{T}(M/G)|_{U/G}$ into $T\mathbb{R}^k$. We have that $d\eta|_{\tilde{z}(\mathbb{R}^k)} \circ d\tilde{z} \circ d\tilde{p}$ is a 1-form in $\check{\Omega}^1(M/G)|_{U/G}$. By Lemma 4.17, this form lifts to a basic form μ' on U. Equation 8 implies that if $\tilde{\mu}$ is basic at $x \in U$, then

$$\tilde{\mu}|_{\check{T}^{\mathcal{A}}_x M} = (\pi|_U)^*(d\eta|_{\tilde{z}(\mathbb{R}^k)} \circ d\tilde{z} \circ d\tilde{p})|_{\check{T}^{\mathcal{A}}_x M}.$$

In particular, if $\tilde{\mu}$ is basic on U to begin with, then $\mu'_U = \tilde{\mu}|_U$. ☐

Proof of Theorem 4.14. We show the surjectivity of π^*. Let $\tilde{\mu}$ be a basic k-form on M. As constructed in the proof of Lemma 4.20, there is a k-form $\mu \in \Omega^k(M/G)$ whose pullback via π is basic, and by Equation 8, is equal to $\tilde{\mu}$.

Together with Lemma 4.18, we have that π^* is an isomorphism of exterior algebras. By Corollary 4.13, since π is orbital with respect to $\mathcal{A}$ and $\mathrm{Vect}(M/G)$, it is in fact an isomorphism of complexes. ☐

Differential Forms on Z and Z/G.

Definition 4.21. Define $\check{\Omega}^k_{basic}(Z)$ to be the linear subspace of $\check{\Omega}^k_{\mathcal{A}_Z}(Z)$ consisting of *basic forms*; that is, forms that are G-invariant and horizontal forms (the latter meaning $v \lrcorner \mu = 0$ for all $v \in \check{T}^{\rho_Z(\mathfrak{g})}Z$).

Lemma 4.22. *Fix $\mu \in \check{\Omega}^k_{\mathcal{A}_Z}(Z)$. Then $\mu \in \check{\Omega}^k_{basic}(Z)$ if and only if there exists $\eta \in \Omega^k_{basic}(M)$ such that $i^*\eta = \mu$.*

Proof. Assume that there exists $\eta \in \Omega^k_{basic}(M)$ such that $i^*\eta = \mu$. Then, for any $g \in G$, we have

$$\begin{aligned} g^*\mu =& g^* i^* \eta \\ =& i^* g^* \eta \\ =& i^* \eta = \mu. \end{aligned}$$

If $z \in Z$ and $v_1, ..., v_k \in \check{T}^{\mathcal{A}_Z}_z Z$ and for some $l = 1, ..., k$ we have $v_l \in \check{T}^{\rho_Z(\mathfrak{g})}Z$, then

$$\mu(v_1, ..., v_k) = \eta(i_* v_1, ..., i_* v_k) = 0$$

since $i_* v_l \in \check{T}^{\rho(\mathfrak{g})}_{i(z)} M = T_{i(z)}(G \cdot i(z))$. Thus, μ is basic.

Conversely, fix $\mu \in \check{\Omega}^k_{basic}(Z)$ and $z \in Z \subseteq M$. Let $\varphi : V \to \tilde{V} \subseteq \mathbb{R}^n$ be a chart on M about z. By Lemma 4.3 we can choose this chart such that there exists $\tilde{\eta} \in \Omega^k(\mathbb{R}^n)$ with the property that

$$(\varphi|_{V\cap Z})^* \tilde{\eta} = \mu|_{V \cap Z}.$$

After shrinking V and multiplying $\varphi^*\tilde{\eta}$ by an appropriate bump function, we extend $\varphi^*\tilde{\eta}$ to a form η_1 acting on all vectors in $\check{T}^{\mathcal{A}}M$.

Let $g_0 = e \in G$ and let $g_i \in G$ for $i = 1, ..., k$ such that the orbit $G \cdot z$ is covered by open sets $g_i \cdot V$. Let $\{\zeta_i\}$ be a partition of unity subordinate to this cover of $G \cdot z$, and define

$$\eta_2 := \sum_{i=0}^{k} \zeta_i (g_i^{-1})^* \eta_1.$$

Let $U' = \bigcup_{i=0}^{k} g_i \cdot V$. For any $x \in U' \cap Z$, and $v_1, ..., v_k \in \check{T}_x^{\mathcal{A}_Z} Z$

$$\eta_2(i_* v_1, ..., i_* v_k) = \mu(v_1, ..., v_k).$$

Hence there is a G-invariant neighbourhood $U \subseteq M$ of $G \cdot z$ such that $\eta_2|_{U \cap Z} = \mu|_{U \cap Z}$. Averaging and applying Proposition 4.16, we may assume that η_2 is a G-invariant form on M.

Next we apply Lemma 4.20 and Theorem 4.14 to obtain a form $\chi \in \check{\Omega}^k(M/G)$ such that $\iota(\eta_2) = \chi$. Now, $\pi^* \chi$ is a basic k-form on M. Also, since η_2 is basic at all $x \in U' \cap Z$, then $\pi^* \chi = \eta_2$ on $U' \cap Z$. Thus, $i^* \pi^* \chi|_{U' \cap Z} = \mu|_{U' \cap Z}$.

Now, cover $Z \subseteq M$ with a locally finite open cover of G-invariant open sets $\{U_\alpha\}$ where for each α, there exists a basic k-form χ_α satisfying $(i^* \chi_\alpha)|_{U_\alpha \cap Z} = \mu|_{U_\alpha \cap Z}$. Add to this cover the G-invariant open set $U_{\alpha_0} := M \smallsetminus Z$, and so we now have an open covering of M. Define $\eta_{\alpha_0} := 0$. Let $\{\zeta_\alpha\}$ be a G-invariant partition of unity subordinate to $\{U_\alpha\}$. Define

$$\eta := \sum_{\alpha} \zeta_\alpha \chi_\alpha.$$

Then η is basic, and $(i^* \eta)|_Z = \mu$. □

Corollary 4.23. *$(\check{\Omega}_{basic}(Z), d)$ forms a subcomplex of $(\check{\Omega}_{\mathcal{A}_Z(Z)}, d)$.*

Proof. Fix $\mu \in \check{\Omega}^k_{basic}(Z)$. By Lemma 4.22, there exists $\eta \in \Omega^k_{basic}(M)$ satisfying $\mu = i^* \eta = i^*(\eta|_{\bigwedge^k_M \check{T}^{\mathcal{A}} M})$. $d\mu \in \check{\Omega}^{k+1}_{\mathcal{A}_Z}(Z)$, but we require this form to be basic. However, since by Lemma 3.75 i is orbital with respect to $\mathcal{A}_Z$ and $\mathcal{A}$, we have

$$d\mu = di^* \eta = i^* d\eta$$

by Corollary 4.13. But by Theorem 2.34, $d\eta \in \Omega^{k+1}_{basic}(M)$. Appealing once more to Lemma 4.22, we are done. □

Theorem 4.24. *The image of*

$$\pi_Z^* : (\check{\Omega}(Z/G), d) \to (\check{\Omega}_{\mathcal{A}_Z}(Z), d)$$

is contained in $(\check{\Omega}_{basic}(Z), d)$.

Proof. Fix $\mu \in \check{\Omega}^k(Z/G)$ and $v_1, ..., v_k \in \check{T}_z^{\mathcal{A}_Z} Z$ for a fixed $z \in Z$. By Lemma 3.77 $\pi_Z^* \mu$ is well-defined. Let $g \in G$. Then

$$g^* \pi_Z^* \mu(v_1, ..., v_k) = \mu(\pi_{Z*} v_1, ..., \pi_{Z*} v_k) = \pi_Z^* \mu(v_1, ..., v_k).$$

If $v_j \in \check{T}_z^{\rho_Z(\mathfrak{g})} Z$ for some $j = 1, ..., k$, then $\pi_{Z*} v_j = 0$, and $\pi_Z^* \mu(v_1, ..., v_k)$ vanishes. Thus $\pi_Z^* \mu$ is basic. By Corollary 4.13, the exterior derivative commutes with π_Z^*, and so we are done. □

Lemma 4.25. *Any form $\mu \in \check{\Omega}^k_{\mathrm{Ham}(Z/G)}(Z/G)$ extends uniquely to a form in $\check{\Omega}^k(Z/G)$, and this extension induces an isomorphism of complexes $(\check{\Omega}_{\mathrm{Ham}(Z/G)}(Z/G), d) \cong (\check{\Omega}(Z/G), d)$.*

Proof. By Proposition 4.7 and Corollary 4.10, it is enough to show that $\mathrm{Ham}(Z/G)$ induces an open dense orbit. Since $(Z/G)_{(K)}$ is open, connected and dense for some $K \leq G$, it is enough to show that for any x and y in $(Z/G)_{(K)}$, there exist $f_1, ..., f_k \in C^\infty(Z/G)$ and $t_1, ..., t_k \in \mathbb{R}$ satisfying

$$y = \exp(t_1 X_{f_1}) \circ ... \circ \exp(t_k X_{f_k})(x). \tag{9}$$

Since $((Z/G)_{(K)}, \omega_{(K)})$ is a symplectic manifold, and Hamiltonian vector fields of Z/G restrict to Hamiltonian vector fields on $(Z/G)_{(K)}$, it is enough to show that $(Z/G)_{(K)} \in \mathcal{O}_{\mathrm{Ham}(Z/G)}$. It is easy to construct compactly supported Hamiltonian vector fields in a fixed Darboux chart whose local flows induce an orbit equal to the chart. Hence, covering $(Z/G)_{(K)}$ with Darboux charts, we get that $(Z/G)_{(K)}$ is the (unique) orbit of compactly supported Hamiltonian vector fields on $(Z/G)_{(K)}$. Note that these compactly supported Hamiltonian vector fields extend to all of Z/G, and so $(Z/G)_{(K)}$ is an orbit of $\mathrm{Ham}(Z/G)$. $\square$

Proof of the Main Theorem. Let K be a subgroup of G such that $Z_{(K)}$ is open and dense in Z.

Lemma 4.26. *For any $\sigma \in \Omega^k(Z/G)$, there exists $\tilde{\sigma} \in \Omega^k_{basic}(M)$ such that $i^*_{(K)}\tilde{\sigma} = \pi^*_{(K)}\sigma$.*

Proof. Fix $\sigma \in \Omega^k(Z/G)$. Recall the definition of a Φ-basic form Definition 2.53. There exists $\sigma_1 \in \Omega^k_\Phi(M)$ such that $i^*_{(K)}\sigma_1 = \pi^*_{(K)}\sigma$. Averaging, we can assume that σ_1 is G-invariant. Since i is orbital with respect to $\mathcal{A}_Z$ and $\mathcal{A}$, orbits in $\mathcal{O}_{\mathcal{A}_Z}$ are contained in orbit-type strata $Z_{(H)}$. By Lemma 2.54, for each $Z_{(H)}$, $i^*_{(H)}\sigma_1$ is basic on $Z_{(H)}$. Thus,

$$\mu := i^*(\sigma_1|_{\bigwedge^k_M \check{T}^{\mathcal{A}}M}) \in \check{\Omega}^k_{basic}(Z).$$

By Lemma 4.22 there exists a basic form $\tilde{\sigma} \in \Omega^k_{basic}(M)$ such that $i^*\tilde{\sigma} = \mu$. Now for each (H), we have $i^*_{(H)}\tilde{\sigma}$ is a basic form on $Z_{(H)}$. In particular, $i^*_{(K)}\tilde{\sigma}$ is basic on $Z_{(K)}$, and so descends to σ on $(Z/G)_{(K)}$. $\square$

Theorem 4.27. *There is an isomorphism of complexes $\varsigma : (\Omega(Z/G), d) \to (\check{\Omega}(Z/G), d)$ such that for any $\sigma \in \Omega^k(Z/G)$, $\varsigma(\sigma)|_{(Z/G)_{(K)}} = \sigma$.*

Proof. Fix $\sigma \in \Omega^k(Z/G)$. Then by Lemma 4.26, there exists $\tilde{\sigma} \in \Omega^k_{basic}(M)$ such that $i^*_{(K)}\tilde{\sigma} = \pi^*_{(K)}\sigma$. By Theorem 4.14, there exists $\tilde{\mu} \in \check{\Omega}^k(M/G)$ such that $\pi^*\tilde{\mu} = \tilde{\sigma}$. By Lemma 3.78, j is orbital with respect to $\mathrm{Ham}(Z/G)$ and $\mathrm{Vect}(M/G)$. Define $\mu := j^*\tilde{\mu} \in \check{\Omega}^k_{\mathrm{Ham}(Z/G)}(Z/G)$. Now since $Z_{(K)}$ is open in Z and is a G-manifold, by Lemma 3.74, we have that $T_z Z_{(K)} = \check{T}^{\mathcal{A}_Z}_z Z$ for all $z \in Z_{(K)}$. Similarly, $T_x(Z/G)_{(K)} = \check{T}^{\mathrm{Ham}(Z/G)}_x(Z/G)$ for all $x \in (Z/G)_{(K)}$. Hence, π_{Z*} is a surjective map between $\check{T}^{\mathcal{A}_Z}_z Z$ and $\check{T}^{\mathrm{Ham}(Z/G)}_{\pi_Z(z)}(Z/G)$ for all $z \in Z_{(K)}$. For $x \in (Z/G)_{(K)}$ and $v_1, ..., v_k \in \check{T}^{\mathrm{Ham}(Z/G)}_x(Z/G)$ with $\tilde{v}_i \in \check{T}^{\mathcal{A}_Z}_z Z$ so that $\pi_Z(z) = x$ and $\pi_{Z*}\tilde{v}_i = v_i$ for each

i,

$$\begin{aligned}\mu(v_1,...,v_k) =&\tilde{\mu}(j_*v_1,...,j_*v_k)\\ =&\tilde{\sigma}(i_*\tilde{v}_1,...,i_*\tilde{v}_k)\\ =&\pi^*_{(K)}\sigma(\tilde{v}_1,...,\tilde{v}_k)\\ =&\sigma(v_1,...,v_k).\end{aligned}$$

Since $(Z/G)_{(K)}$ is open and dense in Z/G, μ is the unique extension of σ to $\check{\Omega}^k_{\mathrm{Ham}(Z/G)}(Z/G)$. We thus have defined a map $\varsigma : \Omega^k(Z/G) \to \check{\Omega}^k_{\mathrm{Ham}(Z/G)}(Z/G)$. It is an exterior algebra homomorphism. We claim it is an isomorphism. To show this, we construct its inverse.

Fix $\mu \in \check{\Omega}^k_{\mathrm{Ham}(Z/G)}(Z/G)$. Then by Lemma 4.25 there exists a unique $\mu' \in \check{\Omega}^k(Z/G)$ such that $\mu'|_{(Z/G)_{(K)}} = \mu|_{(Z/G)_{(K)}}$. By Lemma 3.77 π_Z is orbital with respect to $\mathcal{A}_Z$ and $\mathrm{Vect}(Z/G)$, and by Theorem 4.24 $\pi^*_Z(\check{\Omega}^k(Z/G)) \subseteq \check{\Omega}^k_{basic}(Z)$. Thus, $\pi^*_Z\mu'$ is basic. By Lemma 4.22 there exists $\tilde{\sigma} \in \Omega^k_{basic}(M)$ such that $i^*\tilde{\sigma} = \pi^*_Z\mu'$. We conclude that $\mu|_{(Z/G)_{(K)}} \in \Omega^k(Z/G)$. This is simply the restriction map $|_{(Z/G)_{(K)}}$, which is an exterior algebra homomorphism. Denote this map by ϱ.

It is clear that ς and ϱ are inverses of one another. It remains to show that these are isomorphisms of complexes, that is, that they commute with the exterior derivative d. But for any $\sigma = \varrho(\mu) \in \Omega^k(Z/G)$,

$$d(\varsigma(\sigma)) = d(\varsigma(\varrho(\mu))) = d\mu.$$

But

$$d\mu = \varsigma((d\mu)|_{(Z/G)_{(K)}}) = \varsigma(d(\mu|_{(Z/G)_{(K)}})).$$

Since $\mu|_{(Z/G)_{(K)}} = \varrho(\mu) = \sigma$, this establishes the isomorphism of complexes. Composing ς with the isomorphism of complexes from Lemma 4.25, we have our result. $\square$

We now restate the main theorem. Let G be a compact connected Lie group acting in a Hamiltonian fashion on connected symplectic manifolds (M, ω) and (M', ω'), each with proper momentum maps Φ and Φ', respectively. Let $Z := \Phi^{-1}(0)$ and $Z' := \Phi'^{-1}(0)$.

Main Theorem. *If F is a diffeomorphism from Z/G to Z'/G, then F induces an isomorphism of de Rham complexes $(\Omega(Z/G), d)$ and $(\Omega(Z'/G), d)$.*

Proof of Main Theorem. By Theorem 4.27, $(\Omega(Z/G), d)$ is isomorphic as a complex to $(\check{\Omega}(Z/G), d)$, and similarly $(\Omega(Z'/G), d)$ is isomorphic to $(\check{\Omega}(Z'/G), d)$. Since F is a diffeomorphism from Z/G to Z'/G, it is orbital and so induces an isomorphism of complexes $F^* : (\check{\Omega}(Z'/G), d) \to (\check{\Omega}(Z/G), d)$. Putting these isomorphisms together, we obtain an isomorphism of complexes from $(\Omega(Z'/G), d)$ to $(\Omega(Z/G), d)$. $\square$

5. Reduction in Stages

One application of the above theorem is to reduction in stages without regularity conditions. More details are given in [14], but we review briefly.

Let G_1 and G_2 be compact Lie groups acting on a smooth connected symplectic manifold (M, ω) such that their actions are Hamiltonian, and the actions commute. Then, $G = G_1 \times G_2$ acts on M in a Hamiltonian way, and the corresponding momentum maps $\Phi_1 : M \to \mathfrak{g}_1^*$ and $\Phi_2 : M \to \mathfrak{g}_2^*$ combine to form the momentum map for the G-action: $\Phi = (\Phi_1, \Phi_2) : M \to \mathfrak{g}_1^* \times \mathfrak{g}_2^*$. Note that we can assume that Φ is equivariant, which is equivalent to Φ_1 being G_2-invariant, and Φ_2 being G_1-invariant (these conditions we can obtain via averaging). One can obtain a symplectic quotient in three different ways. The first way is as follows: take $Z_1 := \Phi_1^{-1}(0)$. This is a G_1-space. We obtain a G_2-space, the quotient Z_1/G_1, which is in fact Hamiltonian with respect to the symplectic strata of the quotient (again, see [14]). The corresponding momentum map, denoted Φ_2', is obtained from the restriction of Φ_2 to Z_1, which is G_1-invariant. Thus, we can let $Z_{12} := {\Phi_2'}^{-1}(0)$, and take the corresponding quotient: Z_{12}/G_2.

The second symplectic quotient is obtained by repeating the above procedure, but with G_1, Φ_1, etc., replaced with G_2, Φ_2 and so on. The resulting quotient will be denoted by Z_{21}/G_1. Lastly, we can get a quotient directly from M: let $Z := \Phi^{-1}(0)$. Then Z/G is a smooth stratified space with symplectic strata.

$$\begin{tikzcd}
 & Z_1 \arrow[r] \arrow[d] & M & Z_2 \arrow[l] \arrow[d] & \\
Z_{12} \arrow[r] \arrow[d] & Z_1/G_1 & Z \arrow[u] \arrow[d] & Z_2/G_2 & Z_{21} \arrow[l] \arrow[d] \\
Z_{12}/G_2 \arrow[rr, "\cong"] & & Z/G & & Z_{21}/G_1 \arrow[ll, "\cong"']
\end{tikzcd}$$

Theorem 5.1. *The three symplectic quotients Z_{12}/G_2, Z_{21}/G_1 and Z/G are all diffeomorphic.*

Proof. Lerman-Sjamaar show in [14] that all three spaces are homeomorphic, and their corresponding rings of smooth functions are isomorphic. In other words, these three quotients are diffeomorphic as subcartesian spaces. □

Remark 5.2. In fact, Lerman-Sjamaar show that the three quotients above have isomorphic Poisson algebra structures induced by the G-action.

Corollary 5.3. *The three complexes $(\Omega(Z_{12}/G_2), d)$, $(\Omega(Z_{21}/G_1), d)$ and $(\Omega(Z/G), d)$ are pairwise isomorphic.*

Proof. This is an immediate consequence of Theorem 5.1 and the Main Theorem 4. □

Remark 5.4. The above corollary is a new result. It was previously proved in special cases, as described in Sjamaar's paper [29].

Final Words: While the purpose of defining orbital differential forms on subcartesian spaces in this paper is to prove that the differential forms introduced by Sjamaar on the symplectic quotient are intrinsic, they are interesting objects on their own. They have been specifically tailored for the study of spaces partitioned into orbits of a family of vector fields. However, there are other definitions of differential forms on subcartesian spaces. In particular, those defined by Marshall satisfy a Stokes' theorem and a de Rham theorem (see [17], [18], and [35])). The author suspects that orbital differential forms are isomorphic to the forms of Marshall on subcartesian spaces under mild conditions (*e.g.* the subcartesian space is locally acyclic and locally compact).

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